Performance of Modality Tests on the Existence of Bimodality

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Abstract

The current paper focused on Robertson's and Fryer's (1969) conditions applied to determine the existence of bimodality in the mixture of normals. Best modality test is a pre-requisite to evaluate whether the existing data is unimodal or bimodal. On the basis of the same data, the researchers also assessed bimodality through modality tests. For this purpose, modality tests were compared on the basis of size and power properties designed by Monte Carlo simulations. The results showed that all modality tests were of stable sizes, that is, around the nominal size of 5% on the basis of simulated critical values. In view of power assesment, the Silverman Bandwidth test was found to be the best performer test which was also justified by the real data examples.

Keywords: bimodality, mixture of normals, modality test, powersize

JEL Codes: C0, C1, C2, C6

Introduction

In the existing literature, many complexities were found about the modality. To address the nature of modality in the data, many tests, based on different assumptions and mathematical structures developed in the literature. A few of these studies included Hartigan (1985), Muller and Sawitzki (1991), and Silverman (1981) based on the null hypothesis of unimodality against the alternative of multimodality. Similarly, different studies were carried out to evaluate the performance of modality tests keeping in view the different objectives.

Engelman and Hartigan (1969) investigated the nature of bimodality by dividing the data into two parts with same means in null hypothesis and with the concept to maximize the likelihood ratio, however, most of the times the test failed to detect bimodality. Further, the study of Wolfe (1970) was based on the likelihood ratio test which concluded that for unimodal

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distribution the probability of multi-modes was high and seemed as mixture of normals. Silverman (1981) formulated a monotonically decreasing function based on k-bandwidth when sample size was persistent. Muller and Sawitzki (1991) investigated that in order to determine the k-modes excess, Mass test and Hartigan Dip test showed the same results. Similarly, Bianchi (1997) applied different modality tests on the DGP data of 119 countries for the purpose of inspection and identified k-modes reliably. Contributing further, Chen et al. (2001) applied a “Modified Likelihood Ratio Test” to evaluate modality in the mixture of different models with the help of various parameters. The research concluded that the chances of modality existence were only based on the mixture of unimodal distributions rather than the mixture of some other components. In this connection, their study also introduced a common “Likelihood Ratio Test” for the detection of bimodality. Daniel et al. (2008) used Silverman test and Hartigan Dip test on various considerable distributions and a multimodal was identified for all results after the rectification of asymptotic scales. Arshad et al. (2018) compared modality tests on the basis of stringency criteria and concluded that Silverman test outperforms other tests, however, its performance at large bumps and large sample was not very ideal. Jamal et al. (2020) investigated the size performance of four non-parametric modality tests. It was summarized that the Hartigan Dip test has stable size as compared to other three tests on the basis of simulated critical values, while proportional mass test performs worst. For real data series, Arshad et al. (2019) concluded that the Silverman test performs much better as compared to other tests to detect whether the given series was multimodal, bimodal or unimodal.

In this connection, the current study initially applied Robertson’s and Fryer’s (1969) conditions on the mixture of normals to determine the confirmation of bimodality. Secondly, the study also offered the comparison of four modality tests on the distinct case of data where the parameter values showed bimodality. The assessment of tests was based on size and power properties through Monte Carlo simulations. To check the existence of bimodality for DGP (mixture of normal, that is, N (0, 1) + N (μ_2, σ_2^2)), the researchers detected only those values for which the parameters showed bimodality and ignored the parameter values which represented unimodality.

Section 2 explains modality tests used in the current study to make comparisons; a methodological framework indicating data generating process, simulation design, and bimodality conditions as laid down in section 3. Further, section 4 investigates the findings obtained from simulation results with respect to size and power of tests along with empirically evaluating the simulation results. At the end, section 5 summarized the results of the study.

Comparison of Modality Tests

In the current research, four different tests were used to determine the modality. These tests included Silverman's Bandwidth test, Hartigan Dip test, Proportional Mass Test, and Excess Mass Test. These tests were based on the null hypothesis (H_0) of unimodality against the alternative hypothesis (H_A) of bimodality.

Silverman's Bandwidth Test or Bump Test

This test is also called bump test which depends upon Gaussian kernel density and a small type of window width of a unimodal distribution.

This test is applied for at least two modes in the alternate hypothesis. Silverman (1981) determined the k-critical smoothing parameter called bandwidth as the least one for h_k of the kernel density estimation with k various modes. The testing procedure is given below:

The sample x_i which is from kernel density with un-explained density function 'f'as,

$$\hat{f}(x,h) = \frac{1}{nh}\sum_{i=1}^{n} \mathrm{k}\,(\frac{\mathrm{xi}-\mathrm{x}}{h})$$

Smoothing parameter, which stands equal to 'h' and 'k' is the function of Gaussian kernel. Silverman noticed that as 'h' becomes large, the amount of modes in $\hat{f}(x,h)$ decreases. The test statistic of bump test is given below;

$$\hat{h}_{crit} = \inf\{h : \hat{f}(x,h) \text{has single mode}\}$$

The least smoothing parameter $\hat{h}^1_{crit}$ is vital for one mode and their probability $\widehat{P}$ as;

$$\hat{P} = P(\hat{h}^{1*}_{crit} \geq \hat{h}^1_{crit})$$

When $\widehat{P}$ becomes minimum it ultimately determines significant results for this test. This technique starts from one mode and continues up till the test is ineffective to reject H_0 of 'k' modes.

Hartigan Dip Test

Dip test was designed by Hartigan (1985) which found the presence of bigger difference between the data distribution and specific theoretical distribution of one mode in the data. Let $f(x)$ is density function with 'K' modes; the Cumulative Distribution Function $F(x)$ is curved outside under 'K' and curved inside above 'K'.

Also F is the distribution function and $D(F) = d$ for non-reducing functions 'G'. However,$X_L \leq X_U$, G is the highly outwards curved minorant of $(F + d)$ in limit $(-\infty, X_L)$, now in variable, G has a variable of much higher gradient of (X_L, X_u), G is a very small quantity of inwards curved majorant of $(F - d)$ in (X_L, ∞), so the procedure as follows as;

(i) In the beginning let X_L=X_I, X_u=X_n, D = 0.

(ii) Find the Greatest Convex (outwards curved) Minorant 'g.c.m' G and Least Concave (inwards curved) Majorant 'l.c.m' L for F in $[X_L, X_u]$, consider the values concerning with F are correspondingly $g_1, g_2, \ldots, g_k$ and $l_1, l_2, \ldots, l_m$

(iii) To take d = sup$\left|\mathrm{G}(g_i) - \mathrm{L}(g_i)\right| >$ sup$\left|\mathrm{G}(l_i) - \mathrm{L}(l_i)\right|$ and also the Sup exists at $l_i \leq g_i \leq l_{j+1}$ explain as $x_i^0 = g_i$, $x_u^0 = l_{j+1}$ also.

(iv) Taking d = sup$\left|\mathrm{G}(l_i) - \mathrm{L}(l_i)\right| \geq$ sup$\left|\mathrm{G}(g_i) - \mathrm{L}(g_i)\right|$ and also the Sup exists at $g_i \leq l_i \leq g_{i+1}$explain as $x_i^0 = g_i$, $x_u^0 = l_j$.

(v) When $d \leq D$, finish and put $D(F) = D$

(vi) When $d > D$,

put D =sup$\{D, sup_{x_l \leq x \leq x_l^0}\left|G(x) - F(x)\right|, sup_{x_u^0 \leq x \leq x_u}\left|L(x) - F(x)\right|\}$

(vii) Place $x_u^0 = x_u$, $x_l^0 = x_l$ and go back to (ii).

This study used the same method of Hartigan Dip test for any data series and calculated their size and power for comparison with other modality tests.

Proportional Mass Test

Cavallo and Ringobon (2011) examined modality in the region for a special value which was approximately zero at both sides. The Proportional Mass (PM) test determined the mass of values, changes in the absolute value less than 1%, 2.5%, and 5%. PM test was based on the central tendency point (that is, 0%, mode and mean) of the distribution.

PM test determined the magnitude of unimodality on each sides of the center value which observed the density mass between the boundaries. The mass of the interval (−1%, 1%) was found to be higher than the interval (−5%, 5%).

$$P\ (|\Delta p| \leq 1) \geq P\ (|\Delta p| \leq 5)/5$$

Proportional mass for i= 1 and j= 5 then;

$$PM_{1,5}^{0} = \ln(P\ (|\Delta p| \leq 1))/(\ P\ (|\Delta p| \leq 5)\ /5)$$

Proportional mass on both sides of zero as;

$$PM^{0} = \frac{1}{|z|} \sum_{ij\in z} PM_{ij}$$

'Z' is the set of the combinations of i< j and testing for unimodality on two sides of mode denoted by 'm' as;

$$PM^{m} = \frac{1}{|z|} \sum_{ij\in z} \ln \frac{P\ (|\Delta p - m| \leq i)}{P\ (|\Delta p - m| \leq j/(j\ /i)}$$

For positive value of PM^{m} the results showed that the distribution was unimodal and for negative value the distribution was bimodal.

Excess Mass Test

This test was introduced by Muller and Sawitzki (1991) for multimodality and cluster which remained same to Hartigan Dip test when used for 'm' modes. Excess Mass 'EM' test found the ordinary difference of a related distribution to accessible modal, as often uniform distribution.

They considered a distribution function 'F' with sampling density 'f'. The empirical distribution function was $\widehat{F}$ and 'n' was the sample size drawn from 'F'. Mathematically EM test procedure is as,

$$E_{nm}(\lambda) = Sup_{c_1,\dots,c_n}\left[\sum_{j=1}^{m}(\widehat{F}\ (C) - \ \lambda\left\|C_j\right\|)\right]$$

As $\lambda \geq 0$, the selection of supremum from the set denoted $\{C_1, C_2, \ldots\ldots\ldots, C_m\}$ of disjoint values. The function $\widehat{F}\,(C)$ is equal to $\widehat{F}$ size of C and magnitude $\|C_k\|$ denoted the length of C.

$$D_{nm}(\lambda) = E_{nm}(\lambda) - E_{n.m-1}(\lambda) \geq 0$$

Where H_0 has sampling density 'f' with $(m-1)$ modes and H_1has 'm' modes. The test statistics are as,

$$\Delta_{nm} = \text{Sup}_{\lambda>0}\,\{D_{nm}(\lambda)\}$$

For large value of Δ_{nm} in most cases this test has significant decision. They also introduced empirical procedures for quantity and described the ideas of higher Δ_{nm}, which emphasized that mode 'm=1'.

Methodology

Data Generating Process

In the current study, Data Generating Process (DGP) was used to assess the presence of bimodality with a mixture of two normal distributions (that is, combination of one normal and second one is standard normal distribution). Bimodality conditions and modality tests were applied on the DGP in order to detect the bimodality. The detail of the DGP is explained as:

Here X_1 was the selected sample from the first normal population having location and the scale parameters mean μ_1 and variance σ_1^2 and X_2 was another sample from the second normal population having parameters $\mu_2$and σ_2^2.

M= {X_1 with mixing proportion denoted by p, X_2 with mixing proportion denoted by $(1-p)$}

and, can be written as;

$$M = pX_1 + (1-p)\,X_2 \tag{1}$$

Where "M" denotes a mixture of two normals or also called bimodal distribution and "p" denoted the mixing probability/proportion within interval (0, 1).

Simulation Design for Modality Tests

The current study used the methodology of Monte Carlo simulation design used to determine the four modality tests. The procedure of this simulation design was to find the size and power. These procedures are explained in the following steps:

(i) The DGP in eq. [1] is used to generate the data.
(ii) All the selected modality tests were applied on the same DGP.
(iii) Considering 5% level of significance fixed for Monte Carlo sample size "MCSS"= 5000, then size of the test was calculated as:

Size = Probability (Reject $H_0/\ H_0$ is true)
Mathematically,

$$\text{Size of the test} = \frac{\text{significant count}}{\text{MCSS}} \times 100$$

"H_0" for these modality tests consists of any unimodal distribution, (that is, normal, Chi-square, uniform etc.) and "H_A" contains the DGP.

(iv) Similarly, at 5% level of significance, the power of the modality tests was calculated as:

Power = Probability (Reject $H_0/H_0$is false)

Mathematically,

$$\text{Power of the test} = \frac{\text{Number of rejections out of MCSS}}{\text{MCSS}} \text{ x } 100$$

Bimodality Conditions

The current study used Robertson and Fryer's (1969) conditions with two variables as $X_1 \sim$ N($\mu_1,\ \sigma_1^2$), and $X_2 \sim$ N($\mu_2,\ \sigma_2^2$). The mixture "M" depends on the ratios of parameters, that is, $p,\ \mu = \frac{(\mu_2 - \mu_1)}{\sigma_1}$, and $\sigma = \frac{\sigma_2}{\sigma_1}$ which could be further simplified as $\mu = \mu_2$ and $\sigma = \sigma_2$ due to $\mu_1 = 0$, $\sigma_1^2 = 1$.The parameter values in Robertson and Fryer's (1969) conditions are given as follows:

(i) "M" is called unimodal distribution if $0 < \mu \leq \mu_0$, where

$$\mu_0 = \left\{\frac{2(\sigma^4 - \sigma^2 + 1)^{\frac{3}{2}} - (2\sigma^6 - 3\sigma^4 - 3\sigma^2 + 2)}{\sigma^2}\right\}^{\frac{1}{2}}$$

(ii) If $\mu > \mu_0$ then "M" is called a bimodal distribution also when "p" lies in the interval (p_1, p_2) as $p_1 < p < p_2$.

$$(\sigma^2 - 1)Y_l^3 - \mu(\sigma^2 - 2)Y_l^2 - \mu^2 Y_l + \mu\sigma^2 = 0 \tag{2}$$

Where

$$p_l{}^{-1} = 1+\frac{\sigma^3}{\mu - Y_l}\exp\left\{-\frac{1}{2}Y_l^2 + \frac{1}{2}\left(\frac{Y_l - \mu}{\sigma}\right)^2\right\} \text{ for l= (1, 2).}$$

$$p_1 = \left[1 + \frac{\sigma^3}{\mu - Y_1}\exp\left\{-\frac{1}{2}Y_1^2 + \frac{1}{2}\left(\frac{Y_1 - \mu}{\sigma}\right)^2\right\}\right]^{-1}$$

$$p_2 = \left[1 + \frac{\sigma^3}{\mu - Y_2}\exp\left\{-\frac{1}{2}Y_2^2 + \frac{1}{2}\left(\frac{Y_2 - \mu}{\sigma}\right)^2\right\}\right]^{-1}$$

Where Y_1 and Y_2 are the two roots of eqn [2], with 0< $Y_1 < Y_2 < \mu$, otherwise "M" is unimodal distribution.

(iii)If $\mu \leq$ 2 times minimum of (1, σ), "M" is unimodal distribution. Otherwise $\mu \geq \frac{3\sqrt{3}}{2}$ time minimum of "M" is bimodal distribution for $p_1 < p < p_2$.

It is necessary to get two of the three real roots of (Y_1, Y_2, Y_3) of the 'Y_{th}' cubical eqn (2) and diminishing the complex and negative roots due to the conditions restrictions 0< $Y_1 < Y_2 < \mu$ and set "$p_1 < p < p_2$". However, the researchers calculated the parameters "p, μ_2, and σ_2" values for which the distribution showed bimodality.

After the formatting of this procedure the researchers considered various values of the parameters from the DGP. Looking forward to these parameters values, the current study conducted Monte-Carlo simulations for modality tests and made the assessment based on size and power properties.

Application of Bimodality Conditions

In this section, Robertson and Fryer's (1969) conditions were applied on "DGP" to verify the bimodality existence. As the DGP of this study utilized one standard normal and second one normal distribution so, μ_1= 0, σ_1^2= 1, and varying the values of the remaining parameters (p, μ_2, σ_2^2), that is, p = (0.1, 0.2, 0.3,. . . . . . ., 0.9), μ_2= (1, 2, 3, ... ,10), and σ_2^2 = (0.1, 0.2, 0.3, ... , 0.9). To change the values of these parameters (that is, p, μ_2, σ_2^2) the sample results are shown in the table given below where "1" stands for unimodality while "2" stands for bimodality.

Table 1

Existence of Bimodality Result with the Varying of Parameters in DGP

μ_2	Result where p= 0.1, σ_2= 0.6	μ_2	Result where p= 0.1, σ_2= 0.6
1	1	6	1
2	2	7	1
3	2	8	1
4	2	9	1
5	1	10	1

The Table 1 above describes the bimodality conditions results where μ_2 was measured to be 2, 3, and 4 whereas, p= 0.1 and σ_2^2= 0.6. The DGP was shown to be a bimodal distribution, while for remaining values the DGP remained a unimodal distribution. Similarly, by continuing this process, the researchers obtained the significant conclusions. The results of the DGP are shown in the Table 2 given below,

Table 2

Important Result of the Mixture of Two Normals

When	N(0, 1)+ N(μ_2, σ_2)	N(μ_1, σ_1)+ N(μ_2, σ_2)	Results
$\sigma_2 < \sigma_1$	real and −ve	Real $\pm$ve	1, 2
$\sigma_2 > \sigma_1$	Complex	Complex	2 always

When $\sigma_2 < \sigma_1$ in DGP then eqn (2) gives negative real roots and identified that the distribution was either unimodal or bimodal. However, when $\sigma_2 > \sigma_1$, then the eqn (2) gives complex roots which means that the DGP was unimodal.

When $\sigma_2 < \sigma_1$ in a mixture of two normals (that is, N (μ_1, σ_1^2) + N(μ_2, σ_2^2)) then eqn (2) provided both types of real roots (positive and negative) and detected that the subsequent distribution was either unimodal or bimodal. However, when $\sigma_2 > \sigma_1$ then eqn (2) provided complex roots and it was concluded that the distribution was unimodal. The current study summarized the overall parameter values which help to display bimodality in the following Table 3.

Table 3

Summary of Parameters Values for Bimodality

		μ_2, p, σ_2		
1, 0.1- 0.6, 0.2	2-3, 0.1- 0.4, 0.5	2- 9, 0.1- 0.2,0.7	3- 4, 0.6, 0.7	2- 10, 0.1- 0.4, 0.9
1, 0.1- 0.9, 0.3	2, 0.5- 0.7, 0.5- 0.6	2-7, 0.3, 0.7	2- 10, 0.1- 0.6, 0.8	3- 10, 0.5- 0.8, 0.9
1, 0.5- 0.9, 0.4	2- 3, 0.3- 0.4, 0.6	2- 6, 0.4, 0.7	3- 10, 0.7- 0.8, 0.8	4- 10, 0.9, 0.9
1- 2,0.1- 0.4, 0.4	2- 4,0.1- 0.2, 0.6	2- 5, 0.5, 0.7	5- 10, 0.9, 0.8	

Note. In each cell, first value indicates μ_2, second value indicates p, and third value indicates σ_2. 0.1- 0.6 shows the range (0.1, 0.2, 0.3, ..., 0.6) all other range values can be read in a similar way.

Table 3 indicates the combination of the parameters values, whichprovide the results of the bimodality in DGP to eliminate the other values of the parameters which results for unimodality. These values of the parameters were further used to make bimodal distribution and modality tests to be compared for size and power behaviors.

Results and Discussion

Simulation-Based Comparison of Modality Tests

A Monte Carlo based simulation experiment and empirical comparison of the modality tests was carried out in this section. To perform simulations, various sample sizes (that is, *n* = 60, 120, 220, 350) and Monte Carlo simulation size of 5000 was considered for all tests. Similarly, exchange rates of various countries were considered to empirically evaluate the performance of tests.

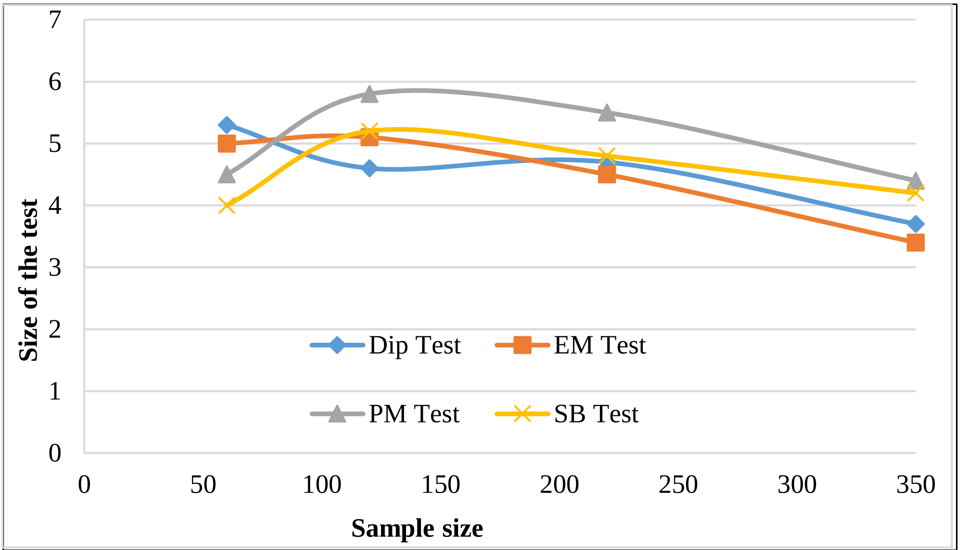
Size of the Modality Tests

In order to test the hypothesis, the null hypothesis (H_0) of unimodality, while the alternate hypothesis (H_A) based on bimodality was formulated. The stable size of all modality tests was calculated through simulated critical values with sample sizes around the nominal size of 5%.

Figure 1 represents the Monte Carlo simulation results of the size of four modality tests with several values of ‘n’. When the sample size is small (*n*= 60), size of all modality tests also varies around 5% of the ominal size with least size of 4.5% for PM test, while being the highest size of 5.3% for Dip test. At sample size n= 120, Dip test exhibits the smallest size (4.6%), while PM test shows the highest size of 5.8%. As the sample size increases from 120 to 220, all four modality tests represent size between 4.5% to 5.5% which are stable theoretically. For the sample size 350, the smallest size (3.7%) was detected corresponding to the Dip test, while PM test had the highest size of 4.4%. Overall, Figure 1 states that all four modality tests have stable sizes around the size of 5%, when decision regarding critical region was taken on the basis of simulated critical values.

Figure 1

Size of the Modality Tests



Power Based Comparison of Modality Tests

This section comparesd the modality tests on the basis of power property with the same sample sizes as has been used to calculate the size in the previous section. Figure 2 to Figure 9 illustrates the power of modality tests with numerous parametric values.

Figure 2 shows the power of four modality tests with parameter values $\mu_{2=}1_{,}p$= 0.6 and σ_2= 0.2. At n= 60, all modality tests have power in between 2.4% to 11.6%, in which PM test has the minimum power of 2.4%. While SB test has 11.6% power and was recognized to be as the most powerful test at small sample size for $\mu_{2=}1_{,}p$= 0.6, and σ_2= 0.2. At n= 120, Dip, EM, and SB tests retained the identical power pattern as has been identified at

n= 60, while power of PM test increased slightly. Furthermore, as the sample size increases, PM test outperforms other tests due to its rapidly increasing power. However, in the remaining three modality tests the SB test has the maximum power as compared to the other two tests at n= 220 and 350. Overall, Figure 2 determines that as the sample size increases, PM test was observed to be as the best performer test with SB test as next best performer test.

Further, a similar picture was observed when μ_2 and σ_2 took the same values as were taken for Figure 2, while the value of p varied (p =0.1, 0.2, 0.3, ... , 0.6). Similarly, when $\mu_2 = 1$, $\sigma_2 =$ 0.4, and p= 0.5, 0.6, ..., 0.9 power results were observed to be nearly equal as observed in Figure 2.

Figure 2

Modality Tests with Power and Parameters (μ_2= 1, p= 0.6, σ_2= 0.2)

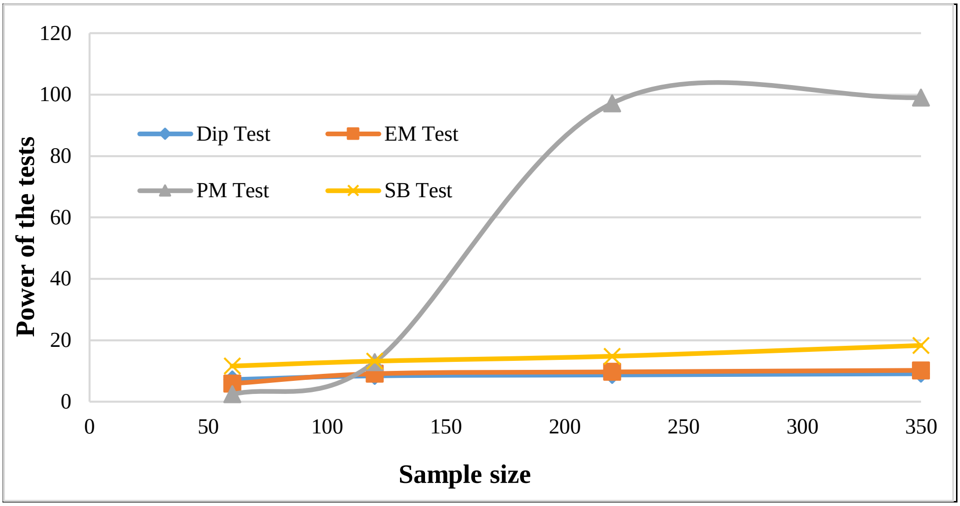


Figure 3 displays the power of modality tests with various parameters, that is, μ_2= 1, p= 0.8 and, σ_2= 0.3. At n= 60 all modality tests gain small power from 2.9% (PM test) to 12.6% (SB test) and were documented as the worst and better performer tests. However, as the sample size increases, PM test outperforms other tests having 54.3%, 60%, and 99% corresponding to n=120, 220 350. Overall, Figure 3 concludes that the PM test was identified as the better performer test, while with small power pattern, Dip and SB tests were recognized as the worst performer tests. In the same way, with the same values of μ_2 and σ_2 , but varying the values of p (p=0.5, 0.6, 0.7, 0.9), all tests showed same results according to Figure 3.

Figure 3

Modality Tests with Power and Parameters (μ_2= 1, p= 0.8, σ_2= 0.3)

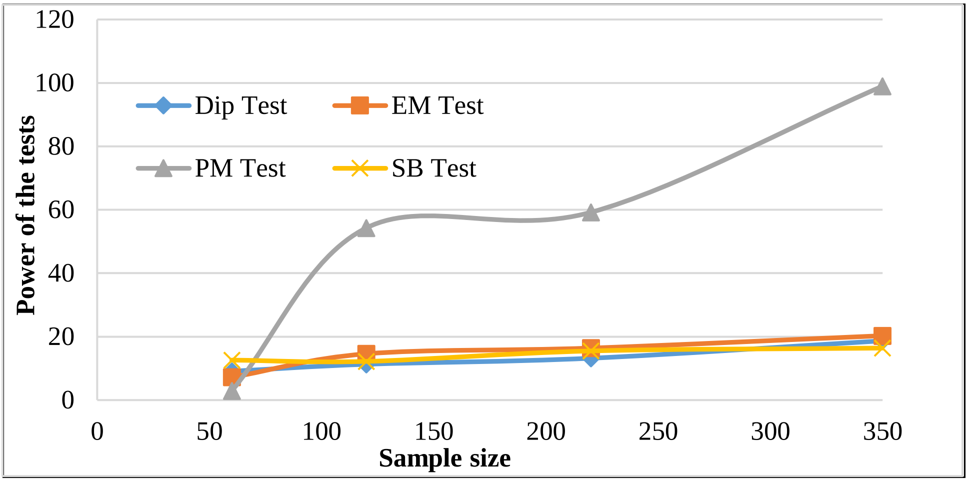


Figure 4

Modality Tests with Power and Parameters (μ_2= 9, p= 0.2, σ_2= 0.7)

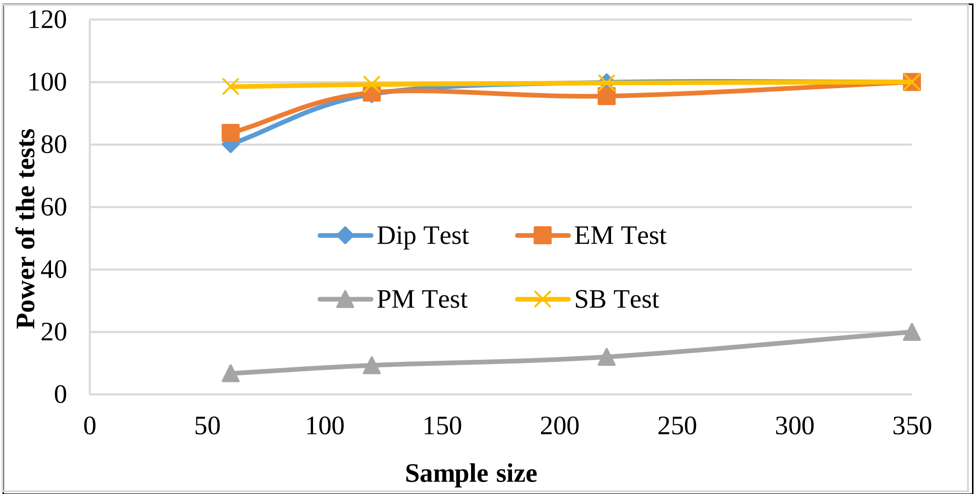


Figure 4 shows the power of modality tests when μ_2= 9, *p*= 0.2 and, σ_2= 0.7. At *n*= 60, Dip with 80.1%, SB with 98.5%, and EM with 83.7% powers outperforms PM test with 6.7% gained power In this situation, SB test was identified to be the best performer test as the small size gets larger. While PM test increased a little in its power pattern and was recognized as the worst performer test. Furthermore, Dip and EM tests with same power behavior over all samples were marked as mediocre performers. Similar results would be obtained if p= 0.2, σ_2= 0.7 are fixed and, μ_2= 7 or 8.

Figure 5 illustrates the power behaviour of modality tests when μ_2= 1, p= 0.6, and σ_2= 0.2. At small sample (n=60), Dip, EM, and SB tests have achieved almost same power close to 98%, while PM tests has the lowest power 43.8%. As the sample size increases Dip, EM, and SB tests achieved maximum power of 100%. While PM test also gained maximum power of 100%, as the sample size increased (that is, greater than n=120). Overall, Figure 5 determines that for small sample size PM test was found as the worse performer, while as the sample size increased all four modality tests achieved maximum power. Similarly, if p =0.5 and the values of other parameters remain fixed, then the results would also remain the same.

Figure 5

Modality Tests with Power And Parameters (μ_2 = 6, p= 0.4, σ_2= 0.7)

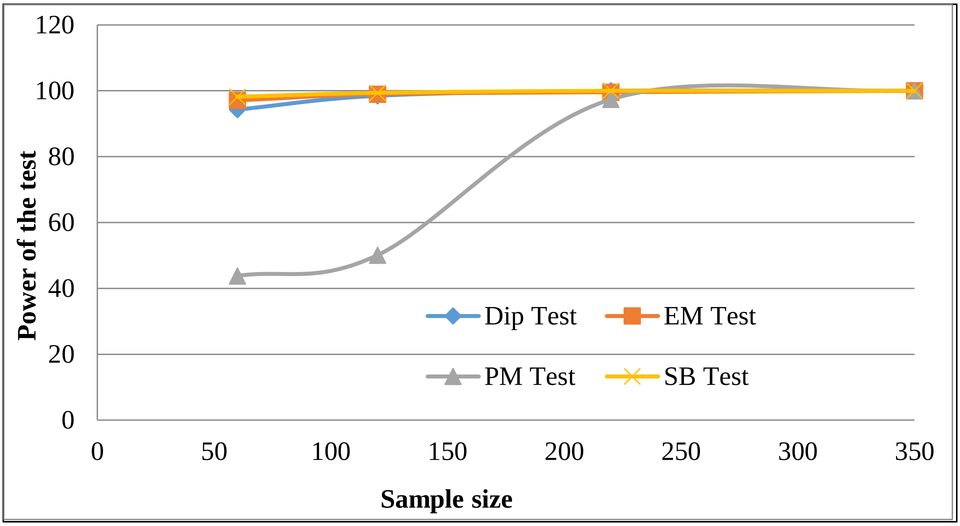


Figure 6 depicts the power of modality tests when μ_2= 8, p= 0.3, and σ_2= 0.8. At n= 60, Dip, EM, and SB tests have obtained the maximum power around 92%-96.6%, while the PM test achieved minimum power (that is, 55.5%). At n= 120, again Dip, EM, and SB tests took maximum power (that is, 98%), while the power of EM test reached up to 72.4%. Furthermore, at large sample sizes Dip, EM, and SB tests gained maximum power of 100%, while PM test gained 97% of its maximum power. Similarly, if p and σ_2 values are kept fixed and μ_2 takes the values of 7 or

9, the results would remain approximately unchanged. The results would also remain the same when $p = 0.7$, $\sigma_2 = 0.8$, and $\mu_2 = 5, 6, 7, 8, 9$.

Figure 6

Modality Tests with Power And Parameters ($\mu_2 = 8$, p= 0.3, σ_2= 0.8)

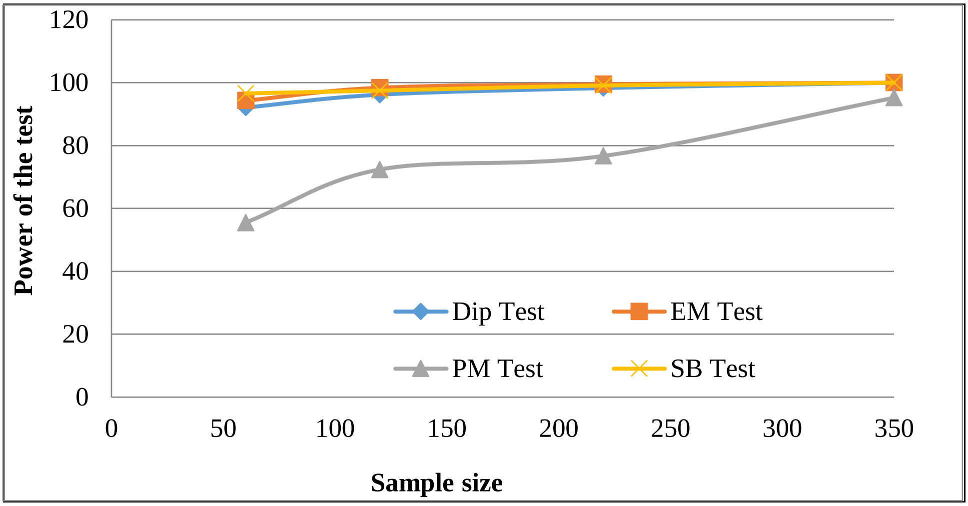


Figure 7

Modality Tests with Power And Parameters ($\mu_2 = 7$, p= 0.9, σ_2= 0.8)

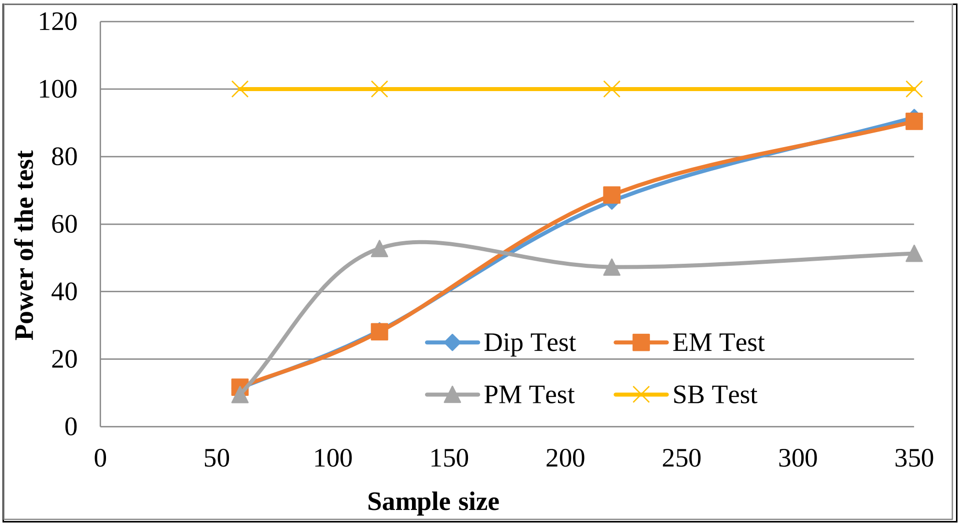


Figure 7 shows power pattern when $\mu_2 = 10$, $p = 0.7$, and $\sigma_2 = 0.8$ in DGP. At $n = 60$, three of the modality tests (that is, Dip, PM and EM) have minimum powers between 10.7% and 15%, while SB test was detected as the better performer with maximum power of 97.5%. When the sample size increased to n= 120, the power of Dip, EM, and PM tests improved with PM test as the much better power gainer. As the sample size increased the power pattern of all tests further improved and Dip and EM tests with same power pattern gained maximum power of 94%. At n= 350, Figure 7 shows that the SB test sustained its supremacy with the highest power of 100%, while power of the remaining three tests also rose, that is, PM with 63%, Dip and EM with 94% individually. Figure 7 concludes that SB test was the best performer test as compared to the other three modality tests, while PM was the worst performer test. Similarly, the results of Figure 7 persist and remain the same, while keeping p= 0.9, σ_2= 0.8, and μ_2=8, 9, 10. The results also remained same when p= 0.9, σ_2= 0.9 are constant, and μ_2= 7, 8, 9.

Figure 8

Modality Tests with Power And Parameters ($\mu_2 = 10$, p= 0.2, σ_2= 0.9)

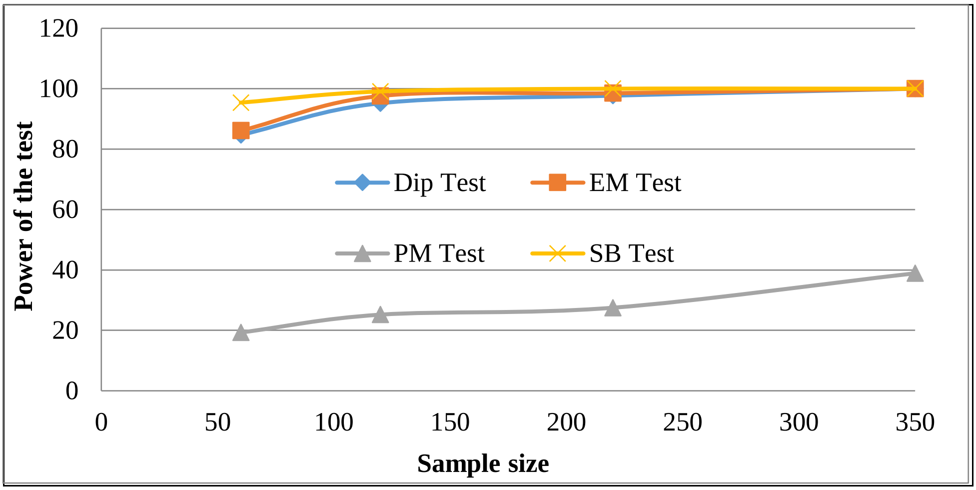
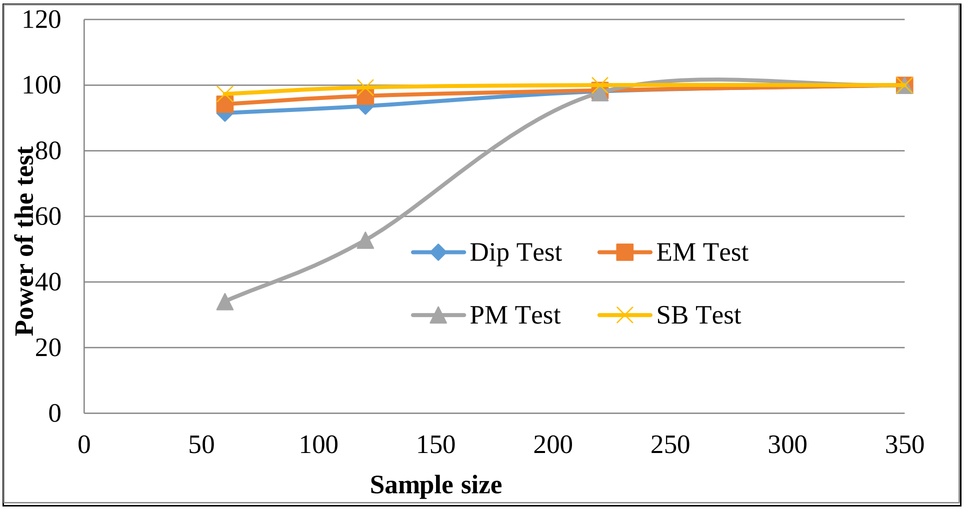


Figure 8 describes the power of modality tests when μ_2= 10, p= 0.2, and σ_2= 0.9. When n= 60, Dip, EM, and SB with 84.7%, 86.1%, and 95.4% have the maximum power attainment, while PM test showed minimum power (that is, 19.3%). As the sample size increased, SB test outperformed other tests and was recognized as the better performer, while PM test was identified as the worst performer with lowest power. Also, it was detected that Dip and EM tests maintain equal pattern of their power attainment at each sample size. Further, if μ_2=8 or 9, p= 0.2 and σ_2= 0.7 then the results remain the same as obtained from Figure 8.

Figure 9 displays power behaviour of modality tests when $\mu_2 = 7, p = 0.6$, and σ_2= 0.9. At n= 60, Dip, EM, and SB tests have attained high power near 98%, while PM test has the lowest power. As "n" increased to 120 then Dip, EM, and SB tests showed almost the same power pattern and a power gain of PM test also increased and reached to 52.8%. At n= 220 and n= 350, Figure 9 shows that all the tests have same power attainment and achieved the maximum power that is 100%. Overall, Figure 9 shows that SB test was the best performer, while at a small sample size PM test was identified as the worst performer among all tests. In the same way, if p= 0.6 and $\sigma_2 =$ 0.9 are kept fixed and μ_2 =5, 6, 8, 9 or 10 then the results would remain the same as obtained from Figure 9.

Figure 9

Modality Tests with Power And Parameters ($\mu_2 = 7$, p= 0.6, σ_2= 0.9)



Overall, keeping in view Figure 2 to Figure 9, it is concluded that SB test performs much better as compared to other tests with respect to power attainment at all specifications. Dip and EM were identified as average performer tests, while PM test as worst performer at all specifications excluding first two situations.

Modality Tests Performance on Real Data

To evaluate the performance of modality tests on real data, annual exchange rates (ER) of Canada, Chile, Colombia, Denmark, Iceland, and Indonesia from year 1976 to 2018 were taken from International Financial Statistics (IFS). Figure 10 shows the density shape of exchange rates which clearly indicates different natures of modality.

Figure 10

Density of Various Exchange Rates

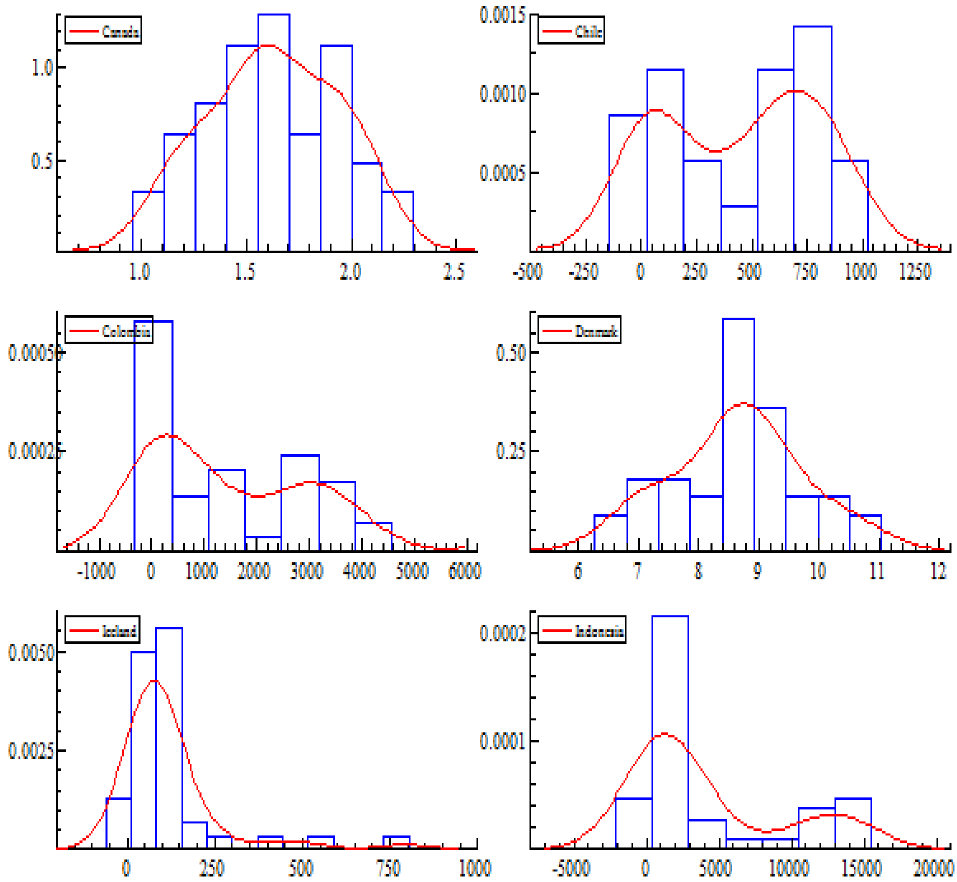


Table 4 shows results of various tests describing whether to accept unimodality or multimodality based on simulated critical values (SCV) to make a decision.

Table 4
Modality Tests Results of Various Countries Exchange Rate

Country	Silverman Test			Dip Test		
	SCV	Calculated value	Decision	SCV	Calculated value	Decision
Canada	0.553	0.4809	1	0.1301	0.0426	1
Chile	0.1263	0.1411	2	0.0741	0.1051	2
Colombia	0.0102	0.0234	2	0.0956	0.0716	1
Denmark	0.0041	0.0028	1	0.2194	0.0304	1
Iceland	0.5729	0.6309	2	0.7362	0.0548	1
Indonesia	0.0002	0.0006	2	0.0461	0.098	2
Country	Excess Mass Test			Proportional Mass Test		
	SCV	Calculated value	Decision	SCV	Calculated value	Decision
Canada	0.0924	0.0853	1	0.9803	0.9357	1
Chile	0.1673	0.2103	2	0.9163	0.8026	1
Colombia	0.1154	0.1433	2	0.5361	0.6109	1
Denmark	0.2057	0.0608	1	0.4513	-0.2614	1
Iceland	0.2461	0.1096	1	1.4725	1.073	1
Indonesia	0.1251	0.1059	1	1.0263	1.073	2

ER of Canada and Denmark showed unimodal pattern on the basis of results, as the calculated value was greater than the SCV, obtained from all modality tests. This result matches with the graphical representation of ERs corresponding to both of the countries. Chile ER was identified as bimodal according to Silverman, Dip, and Excess Mass tests. However, in view of proportional mass test it was observed to be unimodal. In view of Figure 10 Silverman, Dip, and Excess Mass tests have identified the true nature of modality, while proportional mass test was unable to detect the true pattern of Chile ER series. In case of Colombia, Silverman and Excess Mass tests detected ER as bimodal, while Hartigan Dip and Proportional Mass tests showed a unimodal identification of ER series. However, the actual shape of density spotted in Figure 10 shows that ER of Colombia was bimodal which justified the results of Silverman and Excess Mass tests. For Iceland, all modality tests showed that the ER series was unimodal, however, Silverman test described that the series was bimodal. In actual case the density of this series looked bimodal which further clarified how well

Silverman test captures the true nature of ER series. In the last row of Table 4, results of Indonesian ER are displayed where all the tests identified ER series as bimodal, while Excess Mass test results showed that the series was unimodal and corresponding to the density of Figure 10.

Overall results showed that the Silverman test is an appropriate test and performs better as compared to other tests to identify the true nature of modality of the data. These results were parallel to the results obtained from simulation study.

Conclusion

The current study used Robertson's and Fryer's (1969) conditions for the identification of bimodality. The DGP consisted of the mixture of one standard normal X_1~N (0, 1) and second normal distribution X_2~N(μ_2, σ_2^2) with mixing probability "p".

Section (3.1) revealed some important consequences. Firstly, it stated that if $\sigma_2 < \sigma_1$ in the mixture of normals (one standard and other normal), then eqn [2] provides negative real roots and the distributions were identified as unimodal or bimodal. In the second case, if $\sigma_2 > \sigma_1$ then the same eqn [2] determines complex roots and all the remaining results confirmed the unimodality.

If $\sigma_2 < \sigma_1$ in a mixture of two normal, then eqn [2] provides two real roots (that is, positive and negative) which means that the distribution is unimodal or bimodal. Again if $\sigma_2 > \sigma_1$, then eqn [2] determines the complex roots and signals unimodality. Table 4 shows those values of the mixture parameters which display bimodality. Using the same parameter values, the current study calculated the size and power of four modality tests. Under simulated critical values, the size of all modality tests were stable, that is, they were around the nominal size of 5%.

In power assessment, possessing the mean constant (that is, μ_2= 1) and the values of the remaining two parameters led to an increase in the DGP. Therefore, the PM test was found to be the best performing test with high power. However, when the mean increases along with the various values of the two remaining parameters excluding PM test (which has minimum power), all modality tests performs well. For additional variations in these parameters, the SB test showed high power which indicates that the SB test was the most robust and powerful test, while the PM test was observed as the worst performing test. Empirical evaluation of modality tests further clarified that the Silverman test outperformed other tests to capture the true nature of the modality of ER series.

In future research, the assessment of bimodality tests would be done on the basis of mixture of other distributions in the same framework.

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