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Teaching Module

**Creating Optimal Portfolio and the
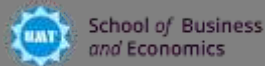Efficient Frontier Using Microsoft
Excel®**

Author(s)
Saurav Roychoudhury[1]

Affiliations
[1]Department of Business, Capital University, Columbus, OH,
USA 43209. Email: sroychou@capital.edu

Teaching Module

Creating Optimal Portfolio and the Efficient Frontier Using Microsoft Excel®

Saurav Roychoudhury[1]

Abstract

Portfolio managers and investors strive to achieve the best possible trade-off between risk and return, and one of the tools they use is constructing mean-variance efficient portfolios. Finance students learn about optimal portfolios and efficient frontiers, though it is difficult to replicate them unless they have access to sophisticated software. This paper develops a teaching module that uses Microsoft Excel® to create mean-variance portfolios and traces out the efficient frontier using real-world data. In the process, the students learn to determine optimal investment allocations in a portfolio, select the optimum investment portfolio given investor's objectives and preferences and learn about factors that influence different asset allocations. For multiple assets (N>3), the paper uses Matrix algebra in Excel®. The paper enables students and investors to learn how to construct real-world mean-variance efficient portfolios using Excel®.

Keywords: Optimal Portfolio, Efficient Frontier, Risk, Expected Return and Risk-free asset.

JEL Classification: G11, A22, A23

[1]Department of Business, Capital University, Columbus, OH, USA 43209.
Email: sroychou@capital.edu

1. Introduction

Finance theory has become increasingly mathematical, but there is a dearth of computational finance materials at the undergraduate level which is likely because undergraduate finance and economics tend to be taught in business schools or under social sciences and more mathematically oriented courses are in mathematics and natural science departments. In recent years we have seen an increase in academic programs that support computational finance but mostly at the graduate level (Roychoudhury, 2007). One of the first papers which laid the groundwork for mathematical theorization in finance was *Portfolio Selection by Markowitz* (1952). The paper introduced the Modern Portfolio Theory (MPT) and formulated the concept of optimal portfolios and the efficient frontier. This paper provides a teaching tool to create mean-variance portfolios and traces out the efficient frontier using real-world data and Microsoft Excel®. In the process, the students learn to determine optimal investment allocations in a portfolio, select the optimum investment portfolio given investor's objectives and preferences and learn about factors that influence different asset allocations.

The finance industry uses proprietary software that runs into thousands of dollars in annual license fees that estimate the efficient frontier and the optimal portfolio. However, Microsoft Excel® has developed into a powerful tool that can be used to model and do sophisticated calculations in finance. For example, see Wann (2015), Boudreaux et al. (2016), and Wann & Lamb (2016). Hess (2005) finds that "hands-on" use of spreadsheet modeling in class, improves understanding and retention of the concepts. Incorporating powerful Excel tools into finance teaching can help students understand the concepts of finance intuitively, and bridge the gap between financial theories and real-world applications (Zhang, 2014). Finance faculty at most leading business schools advocate the use of Excel to prepare students for the workforce.

Investors and portfolio managers concentrate their efforts on achieving the best possible trade-off between risk and return. For portfolios constructed from a fixed set of assets, the risk/return profile varies with the portfolio composition. Portfolios that maximize the return, given the risk, or, conversely, minimize the risk for the given return, are called optimal portfolios. The set of optimal portfolios in

the risk/return plane is called the efficient frontier. In this paper, to create optimal portfolios in Excel and to trace the efficient frontier, we use a dataset of US stocks. The stocks prices are one of the most widely available financial data in the United States. There are several good websites where students can download price information. Some of them are Yahoo! Finance, MSN Money, NASDAQ.com, and Bloomberg.com. In this paper, price information from Yahoo! Finance has been used. For the data and solutions used in this paper, refer to the Excel file (JQM_EF_Excel.xls) which is available for download at http://bit.ly/EFrontier. This paper is organized as follows; the next section is a review of concepts such as expected return, risk, and diversification. This is followed by the section 3 where we construct the portfolio model for 2 risky assets, 3 risky assets and an 'N' number of risky assets. Section 4 implements the model in Excel and section 5 concludes.

2. Overview

2.1. Expected Return

Stock price changes or returns are random variables as the future returns on a stock are uncertain and unpredictable. A stock can have significant 'up' and 'down' movement even within a small time-frame like a single day. Figure 1 shows how the stock price of Microsoft stock fluctuated in a single year. For finding an estimate of future returns on assets such as Microsoft stock, we need to estimate the expected value of the Microsoft stock returns. Ideally, we would try to come up with an expected value of the stock by associating the returns with a probability distribution (very much like the coin-toss example where the probability of getting 'heads' and 'tails' is 50% each).

However, in reality, no model of finance is likely to claim that investors can find great bets "+$1 million with 99% probability" and "−$100 with 1% probability." Such an expected return would be way out of line. In financial markets, no one knows the correct model of expected stock returns well enough to know if the stock market can set the price of the Microsoft stock to offer an expected rate of return on Microsoft of 7% or 12% a year. As it is difficult to estimate true expected returns historical return averages are used as proxies. In this module, we employ the most widely used measure: Expected return

on a stock is the mean return of its historical returns. For example, the expected mean monthly return of Microsoft would be simply

$$E(r_{msft}) = \frac{1}{N} \sum_{i=1}^{N} \bar{r}_{i,msft}$$

where $\bar{r}_{i,msft}$ is the historical monthly return for i^{th} month and N is the number of months over which the returns are averaged.

The expected return of a portfolio, μ_p, is the weighted average of the expected returns on the individual assets in the portfolio, with the weights being the percentage of the total portfolio invested in each asset.

$$E(r_p) = s_A E(r_A) + s_B E(r_B) + K K + s_n E(r_n)$$

$$\mu_p = \sum_{i=1}^{n} s_i \mu_i$$

where $E(r_p) = \mu_i$ is the expected return on the individual stocks, s_i is the weight, and there are n stocks in the portfolio.

2.2. Risk

Statistically, risk measures how dispersed are the outcomes from the center (mean or expected return). Standard deviation is the most common measure of portfolio risk. The higher the level of standard deviation, the more variability between the pay-offs or returns.

Looking back at our example, we can deduce that the variance can be expressed as

$$\sigma^2 = \frac{\Sigma(r - \mu)^2}{N}$$

for N observations and the standard deviation is expressed as the square root of the variance

$$\sigma = \sqrt{\frac{\Sigma(r - \mu)^2}{N}}$$

When we say something like "this investment is risky" we generally refer to the downside risk only. That is, to an investor, the relevant risk of investing in a portfolio is the chance that he

would end up with returns which are less than the expected returns. There is a subtle difference between this and how we measure risk statistically (by variance or standard deviation). When we measure risk statistically, we measure the variability or dispersion around the mean. We put equal weights to the variability of returns both above and below the mean. Though the portfolio expected return is simply the weighted average of the expected returns of the individual assets in the portfolio, the risky-ness (measured by the standard deviation, σ_p) is not the weighted average of the individual assets' standard deviations. The portfolio risk is typically smaller than the average of the standard deviations of the individual assets.

2.3. Diversification

Diversification is equivalent to not putting all eggs in one basket. It is akin to not putting all your money in one risky asset but allocating your money across a number of risky assets.[2] A well-diversified portfolio will significantly lower the downside risk without lowering your expected return (Roychoudhury, 2007). Take the coin-toss example. Suppose the return on $10,000 you have put in a risky stock depends on the flip of a coin. Heads, it quadruples in value (becomes $40,000); tails, you lose $10,000. The expected return is very good at 100%[3]. Unfortunately, the downside risk is terrible – there is a 50% chance that you would lose your $10,000.

As a rule, portfolio risk declines as the number of stocks in the portfolio increases[4]. Thus, careful diversification can create a portfolio that is less risky and earns more on average for the same degree of risk than any single company stock.

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[2]In this paper we are looking at stocks as the risky asset. In the real world, the investor will have other choices like investing in bonds, Bank CDs, real estate, commodities and derivatives.

[3]It is $50\% \times \$40,000 + 50\% \times \$0 = \$20,000$ or 100% return on the initial $10,000 investment.

[4]In practice, investment managers and finance practitioners find that if you hold anywhere between 25 and 30 stocks you can capture most of the diversification benefits. Adding assets beyond a certain number (like 30) does not generate much incremental benefit as far as reducing portfolio risk is concerned.

3. The Model

We know that it is essential to diversify, but it still does not tell you how much of each security you should purchase. How do you determine investment allocations in a portfolio? Where is the best investment portfolio given the investor's objectives and preferences? What would influence different allocations? To answer these questions, we start with a modified version of the Markowitz's (1952) portfolio theory model.

We begin with the simplest example – one period, two assets, and, normally distributed returns. The model assumes that investors are risk-averse, meaning that if there are two assets that offer a same expected return, the investor will prefer the less risky asset. An investor will undertake increased risk only if compensated by higher expected returns. The model also assumes that the investor's risk-reward preference can be explained entirely by expected return and volatility (measured by standard deviation of historical returns). A risk-free asset exists (in the form of US Treasury Bills), and it is possible to borrow and lend money at the risk-free rate. All stocks are perfectly divisible (e.g., it is possible to buy $1/1000^{th}$ of a share) and there are no transaction costs or taxes.

We use the following notations

r_i = the return (sometimes called *rate* of return) on asset i

n = number of available assets

r_f = the return on the risk-free asset

μ_i = mean of the return on asset i

σ_i = the standard deviation of the return on asset i

σ_i^2 = the variance of the return on asset i

σ_{ij} = covariance between the returns on assets i and j

ρ_{ij} = correlation between the returns on assets i and j

s_i = the share of asset i in the portfolio

$r \equiv r_p$ = the (rate of) return of a portfolio

$\mu \equiv \mu_p$ = the mean of the portfolio return

3.1. Two Risky Assets

We assume that there are only 2 risky assets, A and B, available for consideration in an investment portfolio. The portfolio return is given by

$$r = r_A s_A + r_B s_B \qquad (1)$$

The portfolio shares need to add up to one:

$$s_A + s_B = 1 \qquad (2)$$

Taking expectations of (1):

$$E(r) = s_A E(r_A) + s_B E(r_B) \qquad (3)$$

yields the mean portfolio return

$$\mu = \mu_A s_A + \mu_B s_B \qquad (4)$$

Based on equation (4), portfolio variance is

$$\sigma_p^2 = \sigma_A^2 s_A^2 + 2\rho_{AB}\sigma_A\sigma_B s_A s_B + \sigma_B^2 s_B^2 \qquad (5)$$

which simplifies to:

$$\sigma_P = \sqrt{s_A^2\sigma_A^2 + (1-s_A)^2\sigma_B^2 + 2\,s_A(1-s_A)\rho_{A,B}\,\sigma_A\,\sigma_B} \qquad (6)$$

Combining equations (4) and (5), and using (2) to eliminate the portfolio shares, provides the feasible combinations of mean and standard deviation. The **portfolio frontier** is a plot of these feasible combinations of overall portfolio risk and returns.

Combining equations (4), (5), and (2) yields the portfolio frontier for two risky assets:

$$\sigma_p = \sqrt{\frac{1}{(\mu_B - \mu_A)^2}\left[\sigma_1^2(\mu - \mu_B)^2 - 2\rho_{AB}\sigma_A\sigma_B(\mu - \mu_A)(\mu - \mu_B) + \sigma_B^2(\mu - \mu_B)^2\right]}$$

$$(7)$$

The equation represents a **hyperbola** in mean-standard deviation space.

3.2. Three Risky Assets

For a portfolio consisting of three risky assets *A, B, and C,* the return for the portfolio is simply

$\mu = s_A \mu_A + s_B \mu_B + s_C \mu_C$, and the portfolio risk is given as

$$\sigma_P^2 = s_A^2 \sigma_A^2 + s_B^2 \sigma_B^2 + s_C^2 \sigma_C^2 + 2 s_A s_B \, Cov(r_A, r_B) + 2 s_A s_C \, Cov(r_A, r_C) + 2 s_B s_C \, Cov(r_B, r_C)$$
(9)

Please refer to Appendix A1 for a detailed derivation.

3.3. N Risky Assets

As discussed in the background section, as we keep on adding more and riskier assets, the portfolio risk is expected to go down, but at the same time, the math also tends to become messier. We resort to matrix algebra which can represent a lot of data by sorting them into groups or cohorts which are called rectangular *arrays*. A brief refresher of matrix algebra is provided in the mathematical appendix.

We assume here that investors may invest in a total of *n* risky assets and that no risk-free asset exists. Short sales are not restricted. The portfolio frontier, in this case, was rigorously derived by Merton (1972)[5].

Mathematically the portfolio frontier can be found by minimizing portfolio variance subject to a given expected return. This creates an envelope portfolio (Benninga, 2014). The dual of this decision problem does not provide the same solution: Maximizing the expected return subject to a given portfolio variance only produces the upper half of the portfolio frontier. The lower half is dominated as a higher expected return can be found for any possible variance. The upper half of the portfolio frontier obtained in this manner is called the ***efficient frontier*** for apparent reasons. The envelope is the set of all envelope portfolios, and the efficient frontier is the set of all efficient portfolios (Black, 1972). Empirically, if the assumptions leading to mean-variance analysis are justified, we expect that no individual's complete portfolio lies below the efficient frontier.

Consider the following variable definitions:

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[5] The model used in this paper is a simpler version to suit the level of advanced undergraduate students.

$\sum = [\sigma_{ij}]$, represents the *n x n* variance-covariance matrix of the *n* asset returns, where

μ is a *1 x n* column vector of the expected returns, μ_i .

s represents a *1 x n* column vector of the portfolio shares or weights, s_i .

1 represents a *1 x n* column vector of 1's

$\sum^{-1}$ represents the inverse of a matrix $\sum$

The portfolio frontier is found by minimizing portfolio variance subject to a given portfolio mean:

Minimize with respect to *s*:

$$\frac{1}{2} s^T \sum s \tag{10}$$

Subject to the constraints

$$\mu^T s = \mu_p \tag{11}$$

$$1^T s = 1 \tag{12}$$

Thus the portfolio variance is minimized subject to a given expected portfolio return μ_p and given that all portfolio shares add up to 1.

Using the Lagrangian[6] method with multipliers λ and κ for constraints (11) and (12), respectively, produces the following first-order condition:

$$s^T \sum - \lambda \mu^T - \kappa 1^T = 0 \tag{13}$$

Solving for the portfolio weights gives us:

$$s^{T*} = \lambda \mu^T \sum^{-1} + \kappa 1^T \sum^{-1} \tag{14}$$

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[6]For students new to Lagrangian multipliers it might be a good idea to look at an undergraduate Finance and Economics book like Alpha C. Chiang's "Fundamental Methods of Mathematical Economics." A brief overview on Lagrangian multipliers is written by Steuard Jensen and is available at http://www.slimy.com/~steuard/teaching/tutorials/Lagrange.html

As, Σ is positive definite[7] we can conclude that s^{T^*} does minimize the variance and that the solution obtained here for the portfolio shares is unique. In this partial-equilibrium framework, nothing guarantees that all portfolio shares are positive or below one.

Post-multiplying equation (13) by **s** and using constraints (11) and (12) gives:

$$\sigma_p^2 = \lambda \mu_p + \kappa \qquad (15)$$

Post-multiplying equation (18) by μ : and separately by **1** yields the following two equations:

$$\mu_p = \lambda \mu^T \Sigma^{-1} \mu + \kappa 1^T \Sigma^{-1} \mu \qquad (16)$$

$$1 = \lambda \mu^T \Sigma^{-1} 1 + \kappa 1^T \Sigma^{-1} 1 \qquad (17)$$

Define:

$$A = \mu^T \Sigma^{-1} \mu , B = 1^T \Sigma^{-1} 1 , C = 1^T \Sigma^{-1} \mu , \text{ and } D = AB - C^2 \qquad (18)$$

Note that A, B, C, and D are scalars that depend only on the constant parameters of the set of available assets. It is now straightforward to solve for λ and κ from equations (16) and (17):

$$\lambda = \frac{\left(B \mu_p - C \right)}{D} \qquad (19)$$

$$\kappa = \frac{\left(A - C \mu_p \right)}{D} \qquad (20)$$

Plugging (19) and (20) into equation (19) yields an explicit expression of the portfolio frontier:

$$\sigma_p^2 = \frac{\left(B \mu_p^2 - 2 C \mu_p + A \right)}{D} \qquad (21)$$

The portfolio frontier is a hyperbola in mean-standard deviation space as in the case of 2 risky assets. The reason is intuitive:

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[7]For the variance-covariance matrix to have an inverse, it requires that no two assets are perfect substitutes; so that the matrix is not singular.

Any two points on the frontier can be thought of as mutual funds that are individual assets. Taking different combinations of these two assets must trace out a hyperbola based on case with 2 risky assets, but there is no way that this hyperbola can be different from the *n*-asset frontier as it can, at no point, lie to the left of the *n*-asset frontier (or the frontier wouldn't be a true frontier). Thus, once we understand the 2 risky assets case, we can deduce logically that the *n*-asset frontier must be a hyperbola, as well (Roychoudhury, 2007). Based on the above formula it is now easy to find the *minimum variance portfolio* of risky assets. Differentiating equation (21) with respect to μ_P and setting it equal to zero yields $\mu_P = C/B$ so that we obtain $\sigma_p^2 = 1/B$ (after using the definition of D in (22)).

4. Solution Methodology and Implementation

Given a potential set of assets, the efficient frontier can be created by portfolio optimization. Portfolio optimization involves a mathematical procedure called quadratic programming in which two objectives are considered: Maximizing return and minimizing risk. It is called a Quadratic Programming Problem (QPP) as the objective function consists of second-degree terms. The two objectives are considered: (i) Minimize risk given a specific return, or (ii) maximize return for a given level of risk. The QPP for both objectives is generally subjected to the following constraints:

a) The weight of funds invested in different assets must add to unity;

b) There is no short sale provision.

The constraints imposed on the problem are neither exhaustive nor irrevocable. There might be a maximum limit on which an investor can purchase one stock. Similarly, we can modify the short sales constraint to allow short selling.

Portfolios on the mean-standard deviation (or variance) efficient frontier are found by searching for the portfolio with the least variance given some minimum return. Repeating this procedure for many return levels generates the efficient frontier.

The QPP can be solved using constrained optimization techniques involving calculus or by computational algorithms applicable to non-linear programming problems. Of the two

approaches, the non-linear programming is more versatile as it is comfortable handling both equality and inequality constraints.

We start with a simple Excel exercise in tracing a portfolio frontier and then move to the constrained optimization techniques.

Consider two possible investments, say, JP Morgan Chase (JPM) and Oracle Corporation (ORCL). We have about six years or seventy-two months' worth of data[8] obtained from Yahoo! Finance. The returns are arranged in ascending order of dates shown in figure 2.

It is a good idea to start by defining your inputs into *named* Arrays. With Arrays that you do not have to select the entire range every-time, you want to calculate a formula using JPM. For the mean return, you can write =AVERAGE(JPM), for standard deviation[9] we can write it as =STDEVP(JPM), and for a variance, =VARP(JPM) and so on.

Our *objective* is to trace the portfolio frontier in mean-standard deviation space and identify the efficient frontier and the minimum variance portfolio. For simplicity, assume that both stocks can only have positive weights and there is no short-selling.

Step 1: Using AVERAGE(), STDEVP() and VARP(), find the mean, standard deviations and the variance of JPM and ORCL. Also find the correlation coefficient and the covariance between the two assets, JPM and ORCL using the CORREL() and COVAR() functions.

Step 2: Start with any portfolio weights; say your entire money is invested in ORCL. So, the weight of ORCL is 100%, and JPM is 0% (remember the weights must add up to 1).

Step 3: Use equations (4) and (5) from the previous section to compute the portfolio means and portfolio variance. To calculate portfolio standard deviation, simply take the square root of the

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[8]There is no single consensus on how far back you should go to estimate the expected returns and volatility, 5 to 6 years or 60 to 72 months is popular in empirical finance because that is the time frame used to calculate another important variable called the "beta" which is measure 's', a measure of a stock's volatility in relation to the entire market.

[9]Note that we use STDVEP instead of STDEV. The excel function STDEVP refers to the *population* standard deviation whereas STDEV measures the *sample* standard deviation. Similarly we use VARP instead of VAR.

portfolio variance. These portfolio values correspond to the portfolio weights of 0% in JPM is 100% in ORCL.

Step 4: To trace the portfolio frontier we vary the weights of one of the assets, say JPM (the weight of ORCL will automatically vary as their sum has to add up to 1). In our example, we create a column starting from cell B17 (see the screen-shot below) and put weights of JPM in increments of 5% (you can vary weights in smaller or larger increments). Cell C17 corresponds to the portfolio variance formula; Cell D17 is the square root (=SQRT) of the variance in cell C17. Cell E17 gives the portfolio mean using the formula (4) as before. We copy the columns C17:E17 till the row corresponding to JPM's weight of 100%. The screen-shot in figure 3 shows the columns and the formulas used.

Step 5: We now have the data to trace out the frontier. Go to the Chart Wizard or "Insert-Chart." Select the XY (scatter) and then the second option on the right which is "Data points connected by smoothed lines." Select columns corresponding to the Portfolio Risk (standard deviation) and portfolio return as shown in the screen-shot in figure 3. To get a smooth-looking regular shaped frontier, you may have to vary the units of the X and Y axis under chart options.

We have now successfully created a real-world portfolio frontier with two risky assets, JPM and ORCL. From figure 3, we can see that the efficient frontier is traced by the locus of points from "A" to "B." Any portfolio like "C" which lies below the minimum variance portfolio in the picture (or to the south of the minimum variance portfolio) is *dominated* by the minimum variance portfolio and all portfolios which lie to the northeast of it.

How would you interpret the finding? No risk-averse investor should buy portfolio in the region below A or anywhere else, except for points on the efficient frontier traced from A to C. Depending on the risk-taking ability the individual investor can choose between portfolio allocation A (more risk-averse investor) to allocation B (less risk-averse investor). Remember, you can only expect a higher return on the efficient frontier if you are willing to take more risk.

4.1. Finding the Minimum Variance Portfolio (More Accurately)

To find the exact location of the minimum variance portfolio, we solve the QPP using constrained optimization techniques (we can

very well do it after the pain we went through deriving all the formulas). We start with the non-linear programming method using solver and then move to the calculus approach.

4.1.1. Minimum Variance Portfolio using Solver

Excel solver is a powerful tool for optimization[10] and produces targeted results for your models (the technical jargon is "calibrate your model"). [11] Microsoft solver can be added by selecting "Tools-Add-ins" and choosing the "Solver Add-in."

By using the solver, we can calculate the minimum variance portfolio. The screen-shot in figure 4 shows the solver dialog box. In this box, we have asked the solver to minimize the variance in cell B12, by changing the weight of JPM (cell B17) in the portfolio. In order to ensure that the weights of JPM and ORCL are positive we put a non-negative constraint by ensuring the weight of JPM and ORCL's weight are greater than equal to zero. The relation formula 1-B7 in cell C17 corresponding to ORCL's weight ensures that the sum of the two weights does not exceed 1.

Clicking on "Solve" in the solver dialog box gives (see screen-shot below), the minimum variance portfolio with 64.9% invested in JPM and 35.1% in Microsoft. The minimum Variance portfolio corresponding to 7.01% risk (standard deviation) and 0.81% returns matches our result obtained previously in "Tracing a Portfolio Frontier."

4.1.2. Minimum Variance Portfolio using Calculus Method

Recall the section where we derived the minimum variance portfolio for 2 risky assets (all those derivations coming to some use now). Equation (8) gave us the optimal portfolio weight as

$$s_A = \frac{\sigma_B^2 - \rho_{A,B}\,\sigma_A\,\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\,\rho_{A,B}\,\sigma_A\,\sigma_B} = \frac{\sigma_B^2 - Cov(r_A, r_B)}{\sigma_A^2 + \sigma_B^2 - 2\,Cov(r_A, r_B)}$$

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[10]Solver tool uses the Generalized Reduced Gradient (GRG2) nonlinear optimization code developed by Leon Lasdon, University of Texas at Austin, and Allan Waren, Cleveland State University.

[11]A nice book on using excel tools like Solver and Goal-seek and its application in Finance is by Simon Benninga, "Principles of Finance with Excel".

We plug the values of variance and covariances in cell B26 to get the optimal weight of JPM. Implementing the formula in Excel gives the same answer as that given by solver (see figure 5).

4.2. Effect of Change in the Correlation Co-efficient in a Portfolio

Stock A and stock B are two risky assets which have the following characteristics. Stock A has an expected return of 3% but a standard deviation of 25%, while for stock B, the respective numbers are 2% and 15%. The correlation coefficient between the two assets is 0.54. Let us see what happens if the correlation coefficient changes to two extremes; perfectly negatively correlated and perfectly positively correlated.

We trace the portfolio frontier using the same methodology as in "Tracing a Portfolio Frontier" section. We do it for three different values of the correlation coefficient ρ_{AB}. For $\rho_{AB} = 0.54$; $\rho_{AB} = -1$ and for $\rho_{AB} = +1$ as shown in figure 6.

There is a trick to superimpose all the three graphs in the same diagram. Select any graph, copy it and paste it over the other graph, it should clearly superimpose if the size of the graph and the axis are identical. Do the same for the third graph, and we have a graph very similar to figure 3 in the Model section. The screen-shot below (figure 6) of our example is shown below. We can observe that the best diversification can be obtained for $\rho_{AB} = -1$.

For a detailed derivation and explanation of the effect of a change in correlations, refer to Appendix A2.

4.3. Minimum Variance Portfolio for Three Risky Assets

Let us pick a third asset, Haliburton (HAL), which would give us a three asset portfolio with ORCL and JPM. The primary objective is to find the minimum variance portfolio and the weights of the three assets which correspond to the minimum variance portfolio

4.4. Solving the QPP: Minimizing Portfolio Risk Subjected to the Constraints

- The weight of funds invested in different assets must add to unity;

- There is no short sale provision.

We specify the first constraint by ensuring that the weight of an asset is one *minus* the sum of the weights of JPM and HAL. To incorporate the second constraint, we specify that the weights are non-negative in the solver dialog box as shown in figure 7. We set the target cell as B12 which corresponds to the portfolio variance and allow the solver to change the portfolio weights (Cells B17 and C17) to solve for the minimum variance portfolio.

The results (see figure 7) are reflected in cells B11 for expected return on the portfolio, cells B12 and B13 for minimum variance and the minimum standard deviation. The new portfolio weights are displayed in cells B17:D17. The minimum variance portfolio is characterized by an expected return of 0.90% and risk of 6.84% with portfolio weights of 55.8% for JPM, 30.92% for ORCL and 13.28% for HAL.

4.4.1. Using Matrix Algebra

Formulas become lengthy and complicated as more assets are added to the portfolio. Excel has functions which allow us to do basic matrix operations like addition, subtraction, matrix multiplication, inverse, and transpose. We already started the section by naming the data inputs into *arrays*, which is the basic building block of matrix algebra. We name the following matrices as defined in the model section under "N Risky Assets." We can redo the efficient frontier for 3 risky assets and recreate the whole model using Matrix algebra. The results are available in figure 8.

4.4.2. 5 Risky Assets

Refer to the file "Portfolio_optimization_Matrix.xls" for complete solution and details

μ is an *1 x 5* column vector of mean returns of 5 stocks. The array is named "**mu**."

s represents a *1 x 5* column vector of the portfolio shares , the array is named "**s**."

$\Sigma = \left| \sigma_{ij} \right|$, represents the *5 x 5* variance-covariance matrix of the *n* asset returns, the array is named "**Sigma**."

1 represents a *1 x 5* column vector of 1's, named "**i**."

$\sum^{-1}$ represent the inverse of a matrix $\sum$, and can be represented by Excel's matrix inverse function "MINVERSE(**Sigma**)," where "Sigma" is the named variance-covariance matrix.

For matrix multiplication we use "MMULT()," and for transposing matrices, we use the "TRANSPOSE()" function. For details on matrix functions in Excel refer to Excel's Help menu. When you enter a matrix algebra formula in Excel, remember to press CTRL + SHIFT+ ENTER together to execute the formula. Simply pressing ENTER would give an error.

Rewriting equations (10), (11) and (12) we have the corresponding equations (with *) in Excel notation the **QPP** is given as:

Minimize with respect to *s*:

$$\frac{1}{2} s^T \sum s = \text{MMULT(TRANSPOSE(s),MMULT(Sigma,s))} \quad (10^*)$$

Subject to the constraints

$$\mu^T s = \mu_p \quad \{=\text{MMULT(TRANSPOSE(mu),s)}\} \qquad (11^*)$$

$$1^T s = 1 \quad \{=\text{MMULT(TRANSPOSE(I), s)}\} \qquad (12^*)$$

We set the target cell in the Solver dialog box equal to equation (10*) and specify the constraints as in equations (11*) and (12*) in the "Subject to Constraints" box. Select the range of the portfolio weights of the 5 risky assets (F4:F8) as the cells that would be changed by Solver to reach the optimization solution.

The solver solution is shown in figure 9. The optimal portfolio risk (as measured by standard deviation) is 4.79%, and the corresponding portfolio expected monthly return is 0.6%. The weights in the optimal portfolio are also shown in cells F4 to F8.

4.4.3. Using Calculus Method

Refer to the case of "N Risky Assets" in the model section. We start by re-writing the formulas defined in the model section under "N Risky Assets" in Excel form. We continue with the same worksheet in "Portfolio_optimization_Matrix.xls."

$$A = \mu^T \sum^{-1} \mu$$

as{=MMULT(MMULT(TRANSPOSE(mu),MINVERSE(Sigma)),mu)

$B = 1^T \sum^{-1} 1$

as {=MMULT(MMULT(TRANSPOSE(I),MINVERSE(Sigma)),i)}

[12] $C = 1^T \sum^{-1} \mu$

as {=MMULT(MMULT(TRANSPOSE(I),MINVERSE(Sigma)),mu,

and, $D = AB - C^2$

as =A*B-(C.^2)

For the return on the minimum risk portfolio we take $\mu_P = C/B$ as =C./B in Excel notation, and for minimum standard deviation, we take the square-root of

$\sigma_p^2 = 1/B$ in Excel notation as =SQRT(1/B). Figure 10 shows the solution using the calculus method, and it matches the solution by the solver method.

For the equation of the portfolio frontier in the mean-standard deviation space, we take the square-root of equation (21)

$$\sigma_p^2 = \frac{\left(B\mu_p^2 - 2\,C\mu_p + A\right)}{D} \qquad (21)$$

As, =(((B*(E39)^2-(2*C.*(E39))+A)/D)^(1/2)) in Excel notation, where the cell E39 corresponds to a value of the portfolio expected return.

We can trace the portfolio frontier by selecting different values of expected return and calculating the corresponding standard deviation using the above formula. **Note:** In solving using the calculus method we do not consider the assumption anymore that there are no short sale constraints. Figure 10 also shows the graph of the frontier. Refer to the worksheet "Portfolio_optimization_Matrix.xls" for more description on how to select different values portfolio expected returns.

The line tracing the points from the minimum variance portfolio A to B is the efficient frontier. A risk-averse investor will never hold a portfolio which is to the southeast of point A. Any point to the southeast of A such as C would have a corresponding point like

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[12]Excel does not allow using "C" to name arrays. We use "C." in place of "C".

D which has a higher expected return than C with the same portfolio risk. Clearly, point *D* will ***dominate*** point C. The **optimal portfolio** is the portfolio allocation (or points on the efficient frontier) corresponding to the investor's risk preference. More risk-averse investors would choose points on the efficient frontier that is closer to point *A*. Similarly, an investor who is less risk-averse or can handle more risk would choose a portfolio closer to point *B*.

5. Conclusion

Creating optimal portfolios and tracing the efficient frontier is a skill required for a student in finance. Microsoft Excel® has developed into a powerful tool that can be used to model and do sophisticated calculations in finance. This paper is a teaching module that uses Microsoft Excel® to create mean-variance portfolios and traces out the efficient frontier using real-world data. This paper could be used as an integrated part of a computational finance course or a stand-alone component within the typical investments or security analysis and portfolio management course in finance. If the paper suggestions are followed, the students will be able to build a real-world investment portfolio of risky assets using modern portfolio theory techniques. The students should also be equipped to make modifications to the portfolio if some real-world variable changes or if some assumptions of the model are relaxed. The investors could also use this modeling framework to come up with optimal portfolios from a fixed set of risky assets.

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FIGURES

Figure 1: Microsoft Stock Price Movement from Jan 1, 2017, to Dec 20, 2017.

	A	B	C	D	E
1	Date	JPM	_GM		
2	6/1/2001	-0.09517701	0.13093859		
3	7/2/2001	-0.01899631	-0.0114754		
4	8/1/2001	-0.08988439	-0.1324627		
5	9/4/2001	-0.13337567	-0.2164875		
6	10/1/2001	0.04543789	-0.0365965		
7	11/1/2001	0.06694707	0.21684077		
8	12/3/2001	-0.03646518	-0.0221124		
9	1/2/2002	-0.05421071	0.05214153		
10	2/1/2002	-0.14095169	0.04677623		
11	3/1/2002	0.21863198	0.14082126		
12	4/1/2002	-0.00585399	0.06118992		
13	5/1/2002	0.02424662	-0.0239425		
14	6/3/2002	-0.05647616	-0.1400245		
15	7/1/2002	-0.2562724	-0.1290706		
16	8/1/2002	0.05783133	0.04012009		
17	9/3/2002	-0.28063781	-0.1873524		
18	10/1/2002	0.11209626	-0.144979		
19	11/1/2002	0.21298405	0.21148036		
20	12/2/2002	-0.04647887	-0.0716958		
21	1/2/2003	-0.01378631	-0.0144392		
22	2/3/2003	-0.02845731	-0.0572402		
23	3/3/2003	0.04573484	-0.0043368		
24	4/1/2003	0.25552826	0.0722323		
25	5/1/2003	0.11937378	-0.006432		

Figure 2: Time-series Returns of 2 Risky Assets, JP Morgan Chase (JPM) and Oracle (ORCL). Data is on monthly returns from 2001-2007. The data is available from the "Master Data" tab in JQM_EF_Excel.xls and available for download at http://bit.ly/EFrontier.
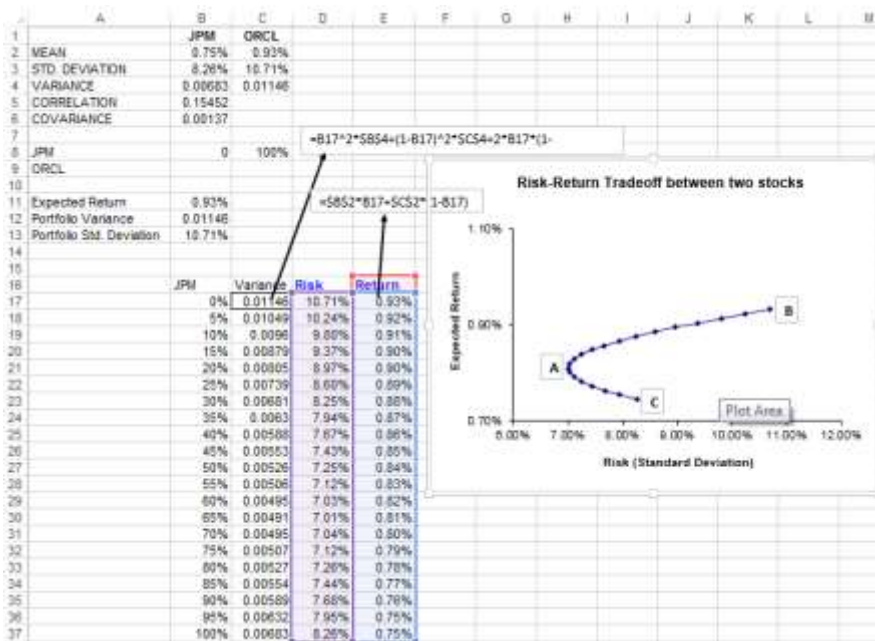
Figure 3: Solving and Graphing for the Efficient Frontier for 2 Risky Assets, JP Morgan Chase (JPM) and Oracle (ORCL). Data is on monthly returns from 2001-2007. The solution is available from the "Tracing Portfolio Frontier" tab in JQM_EF_Excel.xls and available for download at http://bit.ly/EFrontier
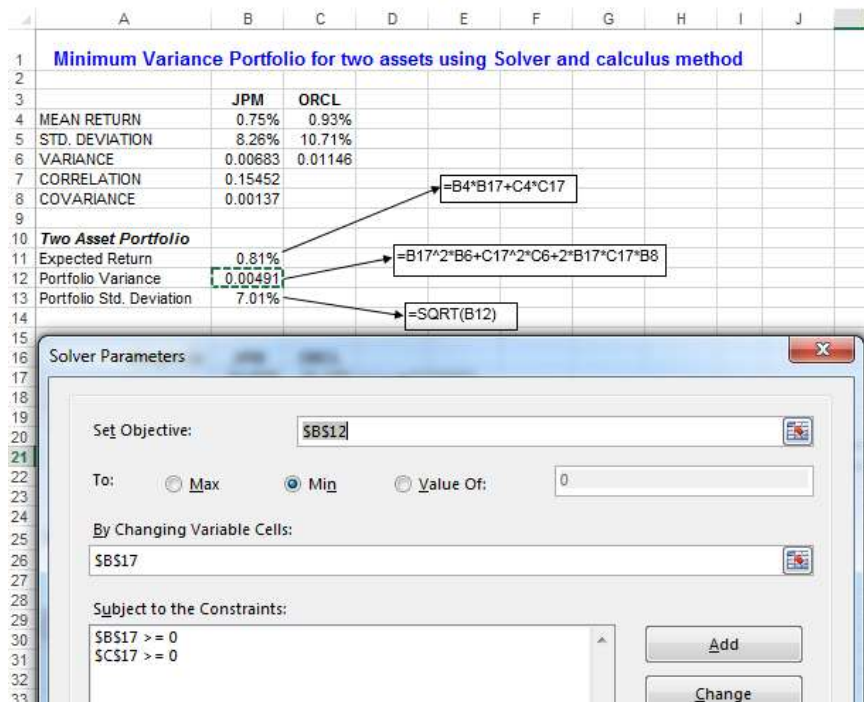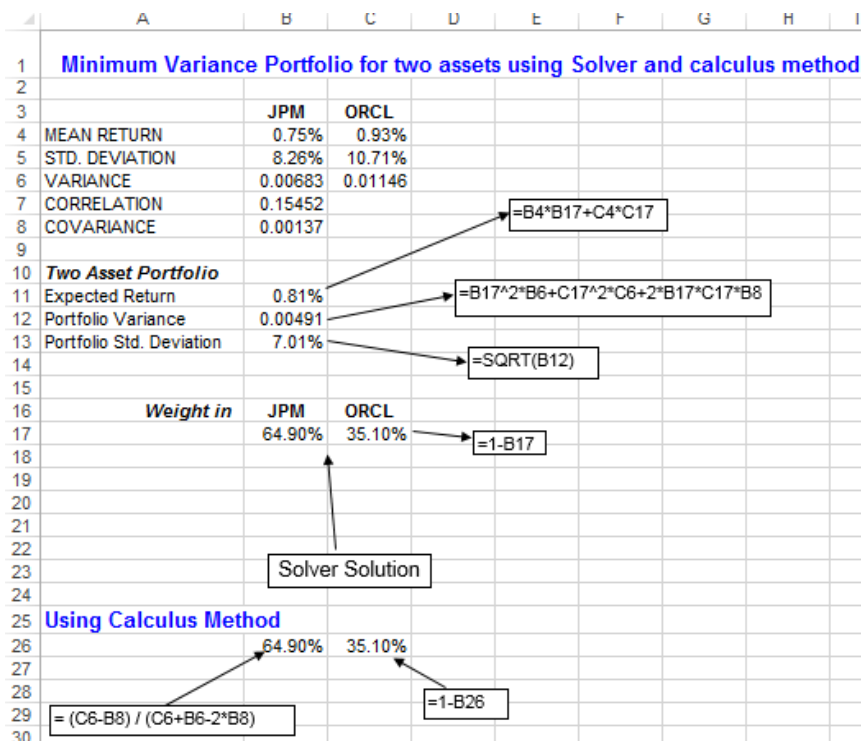
Figure 4: Using Solver tool in Microsoft Excel® to Find the Minimum Variance Portfolio for 2 Risky Assets, JP Morgan Chase (JPM) and Oracle (ORCL). Data is on monthly returns from 2001-2007. The solution is available from the "2 Assets - Solver" tab in JQM_EF_Excel.xls and available for download at http://bit.ly/EFrontier
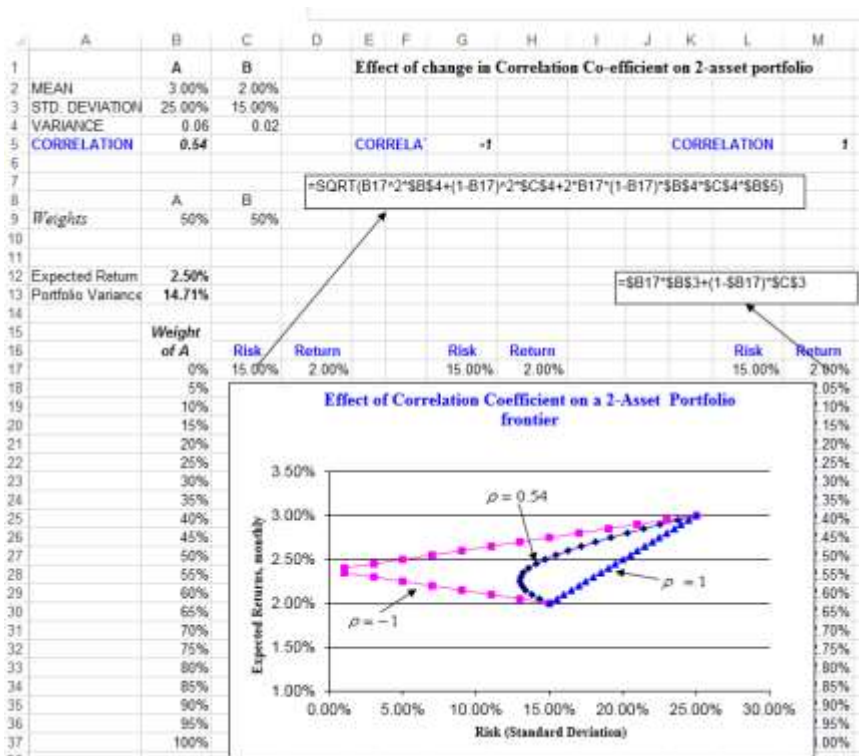
Figure 5: Using the Calculus Method to Find the Minimum Variance Portfolio for 2 Risky Assets, JP Morgan Chase (JPM) and Oracle (ORCL). The figure also shows the solver solution from Figure 5. Data is on monthly returns from 2001-2007. The solution is available from the "2 Assets - Solver" tab in JQM_EF_Excel.xls and available for download at http://bit.ly/EFrontier

Figure 6: Effect of Correlation Co-efficient on Portfolio Frontier for 2 Risky Assets. The solution is available from the "CORRELATIONS" tab in JQM_EF_Excel.xls and available for download at http://bit.ly/EFrontier
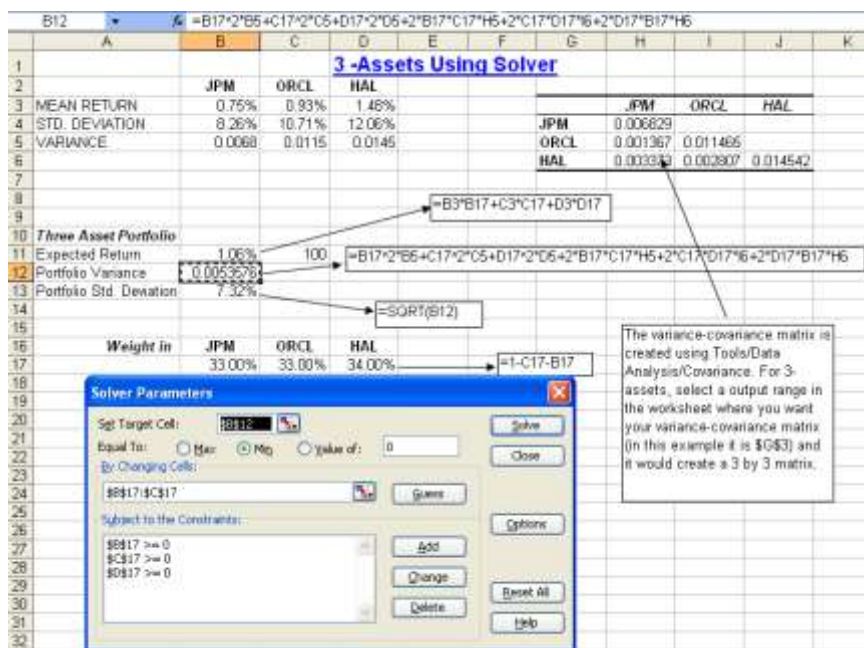
Figure 7: Minimum Variance Portfolio for 3 Risky Assets using Solver. The 3 risky assets are JP Morgan Chase (JPM) , Oracle (ORCL) and Haliburton (HAL). Data is on monthly returns from 2001-2007. The solution is available from the "Minimum Variance – 3 assets" tab in JQM_EF_Excel.xls and available for download at http://bit.ly/EFrontier
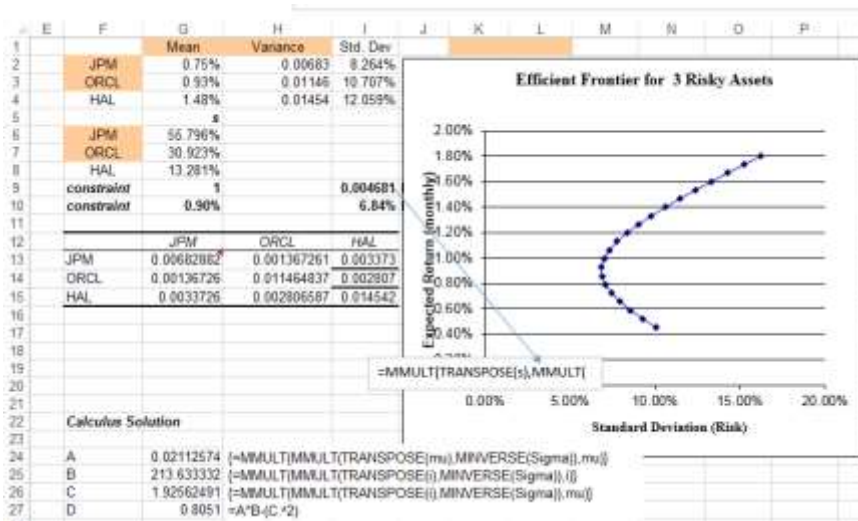
Figure 8: Minimum Variance Portfolio for 3 Risky Assets using Matrix Algebra and Solver. The 3 risky assets are JP Morgan Chase (JPM), Oracle (ORCL) and Haliburton (HAL). Data is on monthly returns from 2001-2007. The solution is available from the "Matrix Algebra – 3 assets" tab in JQM_EF_Excel.xls and available for download at http://bit.ly/EFrontier
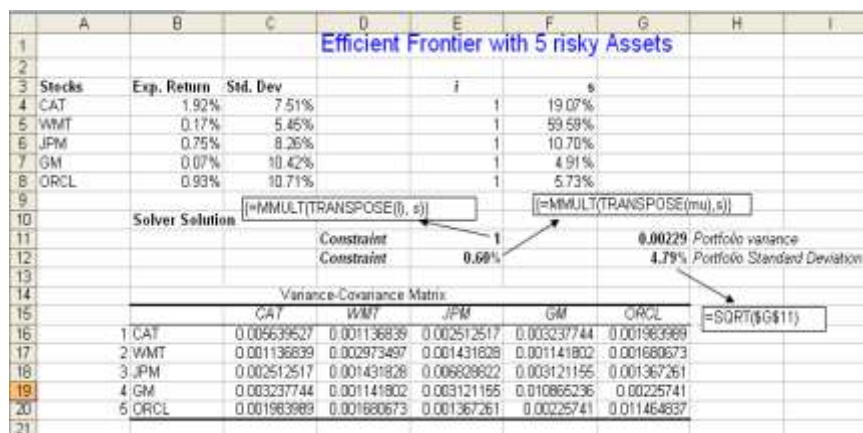

Figure 9: Minimum Variance portfolio for 5 risky assets using Matrix Algebra and Solver. Data is on monthly returns from 2001-2007 in JQM_EF_Excel.xls and available for download at http://bit.ly/EFrontier
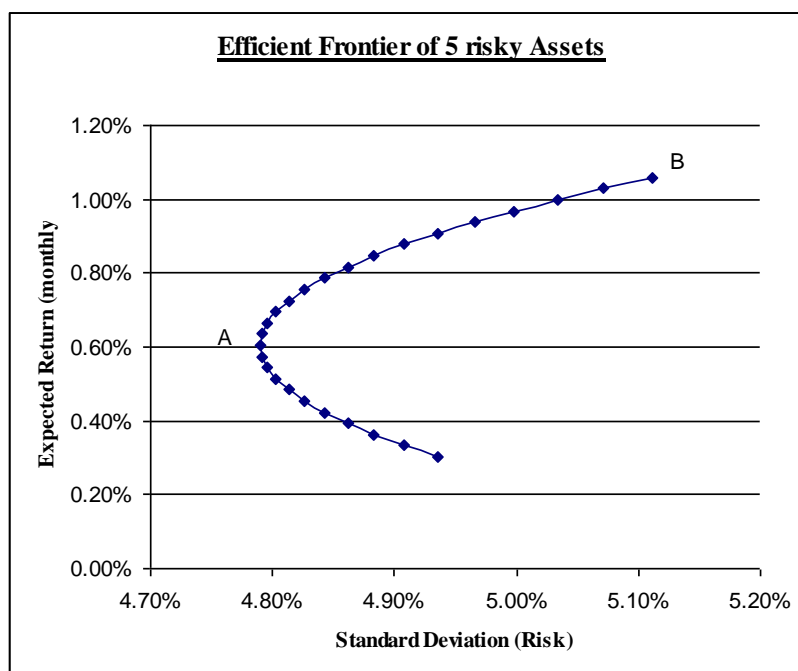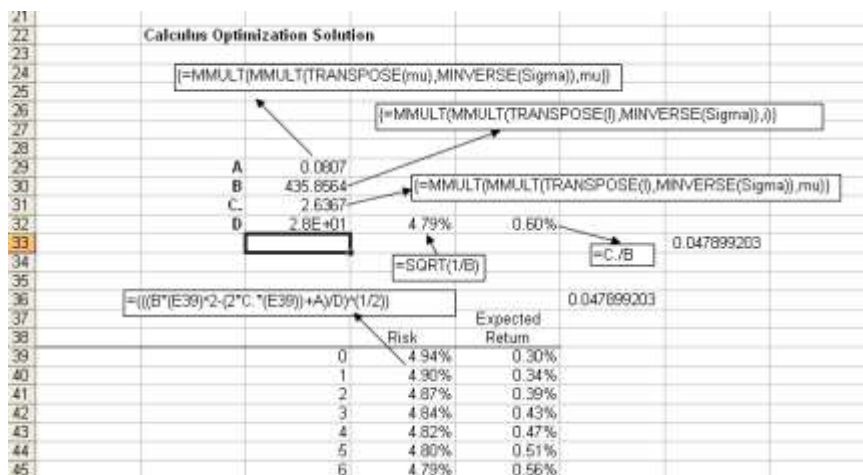
Figure 10: Efficient Frontier for 5 risky Assets using the Calculus Method. Data is on monthly returns from 2001-2007 in JQM_EF_Excel.xls and is available for download at http://bit.ly/EFrontier

APPENDIX A

A1. Derivation of portfolio Expected return and variance for a portfolio consisting of three risky assets *A*, *B* and *C*

The return for the portfolio is simply

$$r = s_A r_A + s_B r_B + s_C r_C$$

Taking expectations on both sides of the above equation we get

$$E(r) = s_A E(r_A) + s_B E(r_B) + s_C E(r_C)$$

This can be re-written as

$$\mu = s_A \mu_A + s_B \mu_B + s_C \mu_C$$

The notations have their usual significance.

The variance of the portfolio can be written as,
$$Var(r) = Var(s_A r_A + s_B r_B + s_C r_C)$$

Let $X = s_A r_A + s_B r_B$; then we can re-write the above equation as[13]

$$Var(r) =$$
$$Var(X, s_C r_C) = Var(X) + 2Cov(X, s_C r_C) + Var(s_C r_C) \quad (A1.1)$$

Take the first term on the left-hand side, and we get

$$Var(X) = Var(s_A r_A + s_B r_B) = Var(s_A r_A) + 2Cov(s_A r_A, s_B r_B) + Var(s_B r_B)$$
$$= s_A^2 \, Var(r_A) + 2s_A s_B \, Cov(r_A, r_B) + s_B^2 \, Var(r_B) \quad (A1.2)$$

Now, take the second term on the right-hand side of (A1.1)
$$2Cov(X, s_C r_C) = 2Cov(s_A r_A + s_B r_B, s_C r_C)$$
Using the property of covariance[14], we can expand the right-hand side of the above expression as
$$= 2s_A s_C \, Cov(r_A, r_C) + 2s_B s_C \, Cov(r_B, r_C) \quad (A1.3)$$

similarly we get the third term of (A1.1) *as*

---

[13] For any X and Y, $Var(X + Y) \equiv Var(X) + 2Cov(X, Y) + Var(Y)$

[14] $Cov(aX_1 + bX_2, Y) = a\,Cov(X_1, Y) + b\,Cov(X_2, Y)$

$$Var\left(s_C r_C\right) = s_C^2 \; Var\left(r_C\right) \tag{A1.4}$$

Let $\sigma_A^2 = Var\left(r_A\right)$, $\sigma_B^2 = Var\left(r_B\right)$ and $\sigma_C^2 = Var\left(r_C\right)$.

Substituting the values of (A1.2), (A1.3) and (A1.4) in (A1.1) we have the portfolio variance for three risky assets

$$\sigma_P^2 = s_A^2\sigma_A^2 + s_B^2\sigma_B^2 + s_C^2\sigma_C^2 + 2s_A s_B \; Cov\left(r_A, r_B\right) + 2s_A s_C \; Cov\left(r_A, r_C\right) + 2s_B s_C \; Cov\left(r_B, r_C\right)$$

A2. Effect of the Correlation Coefficient on the shape of the portfolio frontier

The shape of the portfolio frontier for two risky assets, *A* and *B*, depends on the degree of correlation between the two assets. The correlation between the risky assets is a crucial aspect of any portfolio decision, so to get an idea of the general shapes of the portfolio frontier that are possible, we explicitly consider three extreme assumptions about the correlation between the returns of assets A and B. The corresponding figure for reference is A2.1 at the end of appendix A2. (For a more detailed description see Roychoudhury, 2007 and Benninga, 2014)

$$\rho = 1 \tag{i}$$

Here , the assets are perfectly correlated. Equation (5) now simplifies to:

$$\sigma_p^2 = s_A^2\sigma_A^2 + s_B^2\sigma_B^2 + 2s_A s_B \sigma_A \sigma_B$$
$$\sigma_p = s_A\sigma_A + s_B\sigma_B \tag{A2.1}$$

Figure A2.1 depicts that the portfolio frontier becomes a straight line sloping up from the point where $s_A = 0$ to the point where $s_A = 1$. The dotted lines indicate the opportunities again when short sales are permitted. When the returns on the risky assets are perfectly correlated, no diversification benefits occur and combining the assets will just lead to a linear combination between the extreme positions of putting the whole portfolio in either of the assets.

$$\rho = -1 \tag{ii}$$

The assets are perfectly negatively correlated. Equation (5) becomes:

$$\sigma_p^2 = s_A^2\sigma_A^2 + s_B^2\sigma_B^2 - 2s_A s_B \sigma_A \sigma_B$$

$$\sigma_p = \left| s_A \sigma_A - s_B \sigma_B \right| \tag{A2.2}$$

Note that, strictly speaking, the absolute value should also be taken in equation (A2.1) if short sales are allowed. Diversification benefits are maximal due to the negative correlation between the asset returns.

$$\rho = 0 \tag{iii}$$

The assets are uncorrelated. One may think that this implies that diversification is not possible; in fact, the benefits of diversification are quite clear in this case. It is one of the basic insights necessary to understand portfolio choice. Even though the middle term in equation (5) drops out due to this assumption, the analysis, in this case, is substantially more complex than in the previous two cases. Equation (5) becomes:

$$\sigma_p = \sqrt{\left(s_A^2 \sigma_A^2 + s_B^2 \sigma_B^2 \right)} \tag{A2.3}$$

Consider the mean/standard deviation tradeoff in this case derived from equations (4) and (A2.3). The slope of the frontier can be written as:

$$\frac{d\mu}{d\sigma_p} = \frac{d\mu / ds_A}{d\sigma / ds_A} = \frac{\mu_A - \mu_B}{(s_A \sigma_A^2 - s_B \sigma_B^2) / \sigma_p} \tag{A2.4}$$

If we assume that: $\mu_A > \mu_B$ and $\sigma_A > \sigma_B$, the sign of the slope in the above equation depends on the denominator. It is easy to see in figure A2.1 that at some point the slope is vertical. The portfolio that produces this point is called the minimum variance portfolio. Further, at the fully undiversified point where s1 = 0, the slope must be negative. Thus, starting from this undiversified point; more diversification is beneficial for every investor with mean-variance preferences: Expected return rises while standard deviation falls.

It is clear that the portfolio frontier in case (iii) lies between the frontiers of cases (i) and (ii). It can be shown that this is true for the general case as well. For general correlation between assets 1 and

2, it is also true that the portfolio frontier has the same hyperbolic shape as in case (iii).
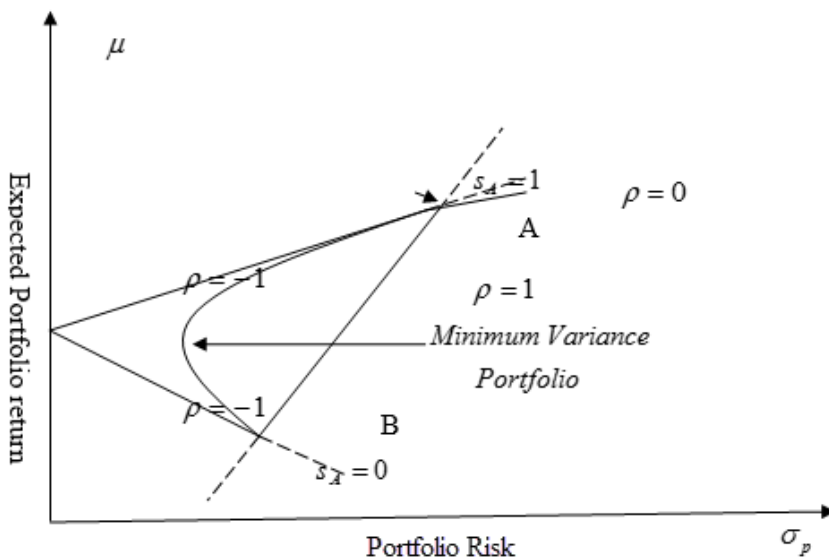

Figure A2.1: Portfolio Frontiers for 2 Risky Assets, A and B. The shape of the portfolio frontier depends on the value of the correlation coefficient between the two assets, A and B.