JQM Journal of Quantitative Methods

Modified Shewhart Control Chart Based on CEV for Gamma Distributed Lifetimes in the Presence of Type-I Censored Data

Author(s)
Syed Muhammad Muslim Raza[1]
Muhammad Moeen[2]

Affiliations
[1] Department of Mathematics and Statistics, Virtual University, Pakistan.
  Email: razamuslim4@gmail.com
[2] School of Business and Economics, University of Management and Technology (UMT), Pakistan.

Modified Shewhart Control Chart Based on CEV for Gamma Distributed Lifetimes in the Presence of Type-I Censored Data

Syed Muhammad Muslim Raza[1]
Muhammad Moeen[2]

Abstract

This article explains the modified version of Shewhart control charts for monitoring the mean level of the Gamma lifetimes under the Type-I censored data. Shewhart control chart based on the conditional expected values (CEV) is developed which can efficiently monitor the Type-I censored. The results of the proposed control chart are compared with simple/traditional Shewhart control chart using different censoring rates (P_c). The main focus is the stability of the mean level for which we have considered the specified parameter(s) as well as the unspecified parameter(s) cases (where Maximum Likelihood Estimates (MLE) has been considered). It is observed that in the presence of Type-I censored observation the CEV Shewhart $\bar{\bar{X}}$ Control chart outperforms traditional Shewhart $\bar{X}$ control chart. The proposed censoring control charts always outperform when known parameters are used rather than the MLE estimate cases. The proposed charting methodology is also illustrated by an example.

Keywords*: conditional expected values (CEV), Gamma lifetimes, outperforms, Shewhart control charts, Type-I censored*

1. Introduction

The manufacturing products are growing swiftly, therefore it is crucial to design and style the products with substantial consistencies. Through the use of the highly censored data accumulated from lifetime

---

[1] Department of Mathematics and Statistics, Virtual University, Pakistan.
 Email: razamuslim4@gmail.com
[2] School of Business and Economics, University of Management and Technology (UMT), Pakistan.

distribution, the valuable time and expenditures can be reduced. A significant factor in life-testing applications for manufacturing technology is exactly how we are able to improve the control chart for tracking the change in mean lifetime of goods whenever the information and facts are censored under Type-I . Lu and Tsai (2008) and Raza, Riaz and Ali (2015) have focused on Type-I censored data using Gamma distribution and also presented EWMA conditional expected values (CEV) control chart for monitoring mean level of the Gamma lifetimes under Type-I censored test.

Steiner and Mackay (2000) designed a one-side charting method using the CEVs that permits for prompt diagnosis of deterioration during this process with seriously censored data under normality. Steiner and Mackay (2001) also proposed the EWMA control charting for efficient monitoring of censored data. Tsai and Lin (2009) proposed EWMA control chart to determine mean changes for the Gompertz distributed lifetimes with the decline and rise in Type-I censoring.

Zhang and Chen (2004) exhibited the realistic performance of censored data evaluation in which they examined a coloring process concerning the rust resilient functionality, scratch panels from a sort of metallic electronic box colored by using this coloring process that are put in a salt spray chamber with the temperature retained at 30° Celsius. It became quite clear that once the data is censored, the simple/traditional control charting just like $\bar{X}$ and R charts shows as large false alarm rates or low power.

Raza et al. (2015) proposed the EWMA control chart to identify the mean shifts in the process mean. They have based their study on the Poisson Exponential distributed lifetimes. Raza, Riaz and Ali (2016) proposed the EWMA and DEWMA control chart to identify the mean shifts for the Poisson Exponential distributed lifetimes with the decrease and increase in Type-I censoring rates.

The Conditional Expected Values (CEV) approach is an effective way to efficiently detect the shifts in the process quality characteristic(s) of interest while dealing the data with high censoring rates, (Steiner & Mackay, 2000). The Shewhart-type control charts make the decision of process stability based on only the most recent

information (Shewhart, 1931), while the Exponentially Weighted Moving Average (EWMA) control charts (Roberts, 1959) are based on past information along with current. Simple/traditional Control charting methodology couldn't deliver a simple yet effective analysis in the existence of censored data therefore to rectify this problem, the statistician /researchers have formulated various procedures for distinct lifetime distributions. Within this paper a technique is discussed which gives an effective result in the existence of censored data for the Gamma distribution.

This paper is structured in the following segments: The introduction is presented in Section 1. Section 2, presents procedure and details. Section 3 presents the numerical computations for conditional expected values (CEVs) and charting tools. Section 4 addresses certain concluding comments regarding the suggested approaches.

2. The Proposed Shewhart CEV Control Charts

There are many real life situations where we are concerned with the life phenomenon like lifetime of different manufactured items, say X (a random variable of interest). In certain practical scenarios this random characteristics X is Gamma distributed (e.g. in reliability modeling, industrial engineering, survival studies etc.) and censoring cannot be avoided (Steiner & Mackay, 2000, 2001). The probability model of a Gamma random variable X is given by:

$$f(x, \alpha, \beta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-x\beta}}{\Gamma(\alpha)} \quad for \ x > 0 \ and \ \alpha, \beta > 0$$

We assume that the life times of the components of interest are denoted by $X_1, X_2 ..... X_n$. Also the Type-I censoring environment is considered for all the items and their lifetimes are reported as $T_{i1}, T_{i2} ..... T_{in}, i=1, 2..., \delta$ if they meet the criteria of presents censoring time say C. It is to be noted that δ represents all under consideration subgroups and/or samples which may vary from situation to situation. Further details in this regard may be seen in Lu and Tsai (2008) and Tsai and Lin (2009). The censoring rate is defined as $P_c = 1 - F(t; \alpha, \beta)$, where $F(t; \alpha, \beta)$ is the cumulative density function of Gamma distribution and is given as

$P(T \leq t) = \gamma(\alpha,\beta x)/\Gamma(\alpha)$ whereas $\gamma(.)$ shows lower incomplete Gamma function.

Now the aim of this research is to monitor the mean level of the censored lifetime under different process situations. Let the mean of Gamma random variable is denoted by μ then it is given as: $\mu = E(x) = \alpha\beta$. For our study purposes if the process is under the state of statistical control, the mean lifetime is assumed to be $\mu_o = \alpha o \beta o$. To monitor the mean level, we may have two cases namely whether α, β are known or unknown. We explore the case when the β is unknown and α known for our study purposes. For the case when process parameter β is unknown we consider the usual Maximum Likelihood Estimator (MLE) denoted by $\hat{\beta}_m$.

2.1. Estimation of $\hat{\beta}$

Let us consider the case where the scale parameter of Gamma distribution is unknown so we compute the estimate (MLE) of the scale parameter of Gamma distribution. Then using these estimates (or the specified parameter for the known parameter case) the samples are generated form Gamma distribution and all items are tested under Type-I censoring setup.

MLE: Using likelihood function

$$L(.) = \prod_{i=1}^{n} [\frac{x^{\alpha-1}}{\Gamma(\alpha)} \beta^{\alpha} e^{-x\beta}]^{\phi_i} \left[1 - (\frac{\gamma(\alpha, \beta c)}{\Gamma(\alpha)})\right]^{(1-\phi_i)}$$

The MLE under Type -I censoring is not easy to calculate and does not have a closed form so MLE's will be obtain for particular data under study that satisfy the following equation:

$$\frac{\partial \ln L(.)}{\partial \beta} = \sum_{i=1}^{n} \phi_i x_i + \frac{n - \sum_{i=1}^{n} \phi_i}{P_c} \left[\sum_{i=\alpha}^{\infty} \frac{x^i [i\beta^{i-1} e^{-x\beta} - x e^{-x\beta} \beta^i]}{i!} \right] = 0$$

where n shows the number of sampling units, X_i (i=1,2,3,…,n) shows the lifetime from Gamma distribution where it is observed when $x_i < C$ therefore, $\phi_i = 1$ if $x_i \leq C$, and $\phi_i = 0$ if $x_i > C$ and

$$\sum_{i=1}^{n} \phi_i = r .$$

The *CEV* for Gamma distribution is calculated as:

Conditinal Expected value $(CEV) = E(T|T > c)$

$$E(T|T > c) = \frac{1}{(1 - F(c,\alpha))} \int_{c}^{\infty} t(\frac{\beta^{\alpha} t^{\alpha-1} e^{-\beta t}}{\Gamma(\alpha)}) dt$$

Where $f(t) = \dfrac{\beta^{\alpha} \ t^{\alpha-1} e^{-\beta t}}{\Gamma(\alpha)}$

$$E(T|T > c) \quad = \left[\alpha_0 \beta_0 \Big/ \sum_{j=0}^{\beta-1} (c/\alpha_0)^j / j! \right] \left[\{\sum_{j=0}^{\beta-1} (c/\alpha_0)^j / j!\} - \{(c/\alpha_0)^{\beta_0} / \beta_0 !\} \right]$$

where c is the censoring time and α is the shape and β is the process (scale) parameter of the Gamma distribution. Now we transfer the Type-I censored data set to:

$$W_{ij} = \begin{cases} T_{ij}, \ if \ T_{ij} \leq C \ (uncensored) \ . \\ CEV(T_{ij)}, if \ T_{ij} > C \ (censored) \ . \end{cases}$$

j = 1, 2, 3,…, n, i = 1,2,3,…, m.

Now working with the transformed distribution formulates, we get the one-sided CEV based mean-$\bar{X}$ chart obtained by plotting the subgroup averages and Standard deviation-S chart by plotting the standard deviations of the subgroups. The Control Limits of the proposed control chart is given as:

$$UCL : \bar{\bar{X}} + L\sigma_{\bar{X}}$$
$$CL : \bar{\bar{X}}$$
$$LCL : \bar{\bar{X}} - L\sigma_{\bar{X}}$$

whereas L is 99.73 quartile points of the Gamma distribution and its average value is calculated as 24.

2.2.   Algorithm for Proposed Control Charts

- Generate data of size n from the Gamma distribution (with the specified parameters) for the k subgroups.
- Now specify the censoring rate Pc and time C.
- Calculate the CEV value and replace the censored observations from the CEV value.
- Now calculate the (UCL, CL, LCL) control limits for the Censored Shewhart control charts;
- Now plot the plotting statistics values against the subgroup numbers and see where it first time fall outside *UCL* or *LCL* respectively and note down the corresponding subgroup/sample number.
- Repeat the steps given above say 5000 times for in-control process and then calculate the average value of 5000 iterated value to get the ARL_o.
- For the proposed control limits first obtain desired ARL_o and then investigate for the ARL_1 performance for shifts in the process scale parameter.

3. Numerical Study and Example

The results have been computed using Gamma distribution with parameters $\alpha = 0.5$ and $\beta = 1$:

Table 1: ARL$_1$ for the Censored $\bar{X}$ Chart Using 30% Shift in Mean (Decrease)

	Shift: 30% Decrease in Mean	
Pc	**ML Estimator** ($\alpha = 0.5$ and $\hat{\beta} = 0.941$)	**Known Parameter** ($\alpha = 0.5$ and $\beta = 1$)
0.2	47	35
0.3	39	30
0.5	35	27
0.6	31	20

Table 1 indicates that the out-of-control ARLs (ARL$_1$) for the proposed CEV $\bar{X}$ control chart (for known parameters and MLE estimates) using different censoring rates. It is observed that the performance of the proposed CEV $\bar{X}$ control chart is preferable for high censoring rates. For the analysis purpose we have used the samples of size n=3 from Gamma distribution.

Table 2: ARL$_1$ for the Censored S chart using 30% shift in Mean (Increase)

	Shift: 30% increase in Mean	
Pc	ML Estimator ($\alpha = 0.5$ and $\hat{\beta} = 0.941$)	Known Parameter ($\alpha = 0.5$ and $\beta = 1$)
0.2	51	45
0.3	46	41
0.5	40	37
0.6	37	32

The Table 2 shows that the out-of-control ARLs for the proposed Shewhart S control charts (for known parameters and MLE estimates) with different censoring rates. It is observed that the performance of the proposed CEV S control chart is preferable for high censoring rates. Table 3 enumerated below reveals that CEV control chart performs appropriately as compared with the conventional control charts.

Table 3: Comparison of ARL$_1$ for Censored and Traditional Shewhart Charts

Pc	CEV $\bar{X}$ Chart (30% decrease in Mean)	Traditional $\bar{X}$ Chart (ignoring censoring)
0.2	35	48
0.3	30	53
0.5	27	61
0.6	20	66

Likewise Table 4 provided below demonstrates CEV S control chart carries out appropriately as compared to the conventional control charts in the existence of censoring data.

Additionally it is observed that for excessive censoring rates the results get more preferable for proposed Censoring control charts.

Table 4: ARL's (out of control) for Censored and Traditional Shewhart Charts

| Pc	CEV S Chart (30% increase in Mean)	Traditional S Chart (ignoring censoring)
0.2	45	56
0.3	41	62
0.5	37	71
0.6	32	78

Observation from Tables 1-4:

- It is quite efficient at detecting shifts in process location parameter and its performance keeps increasing with the decrease in censoring rate for traditional charts whereas for censored chart the pattern is reverse. .
- Its ARL_1 ability keeps improving (gets smaller and smaller) with an increase in the absolute value of Pc and never exceeds ARL_0 value, so there is no issue of ARL biasedness and non-monotonicity as obvious from Tables 1 and 2 (Acosta-Mejia, C. A., 1998; Antzoulakos & Rakitzis, 2010).
- The censoring control charts always performs better than traditional existing Shewhart chart in the presence of censored data.The censoring control charts always outperform when known parameters are used rather than the MLE estimate cases (Table 1 and 2).

The figure given below shows the performance of proposed censoring control chart to traditional Shewhart control chart. Figure 1 and 2 shows that censored control chart perform better than traditional control charts.
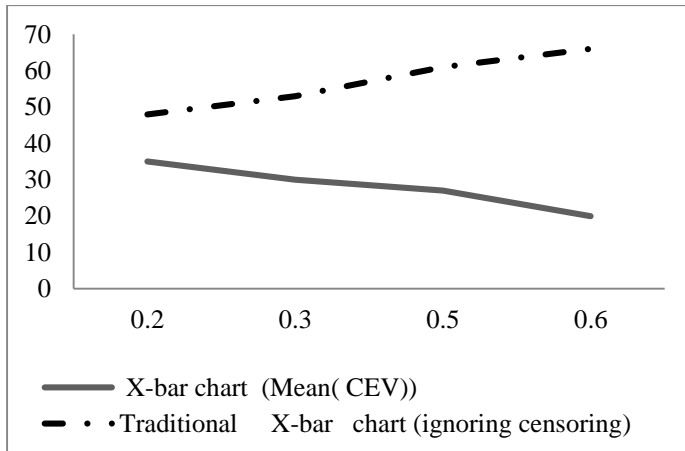
Figure 1: Plot of CEV $\bar{X}$ Chart vs. Traditional $\bar{X}$ Chart (Ignoring Censoring)
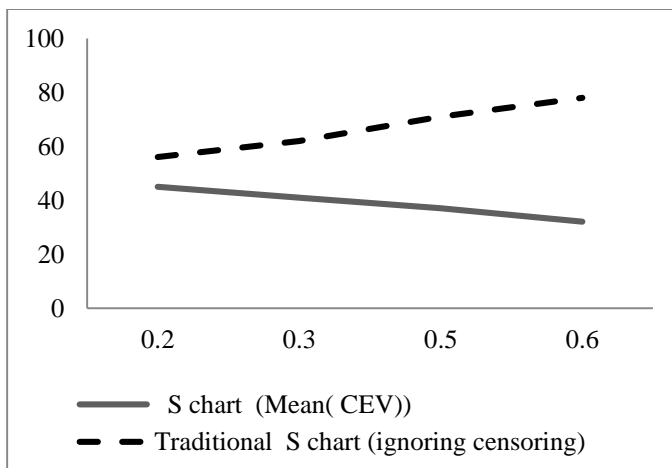

Figure 2: Plot of CEV S Chart vs. Traditional S Chart

4. Illustrative Example

For exploring the proposed control chart we have used a data set from a Gamma distribution. We have taken into consideration $\alpha = 0.5 \,\&\, \beta = 0.941$ that has been the estimated value of unspecified parameter by as well as discussed in section 2. The reason for this selection will be to exemplify and evaluate the CEV Shewhart improvements with conventional control charts for the

identification of variations in process parameter with 25% censoring level for practical application rationale. For that reason, we generated the initial 70 observations from Gamma distribution and those values were regarded as the in-control observation.

The subsequently 48 observations are generated by taking into account the out-of-control, i.e., introducing a shift, situation with 30% decrease in the mean. The sample of size used to generate data (given in Appendix A) and ARL_o=70. The control chart graphic presentation is presented in Figure 3. From Figure 3, it is possible to see that initial 70 data points don't accumulate any alarm as these were generated from an in-control process. Nevertheless, after the 70th subgroup, we get two out-of-control signals. The first out of control signal is detected at 103 point which shows the $ARL_1 = 33$.
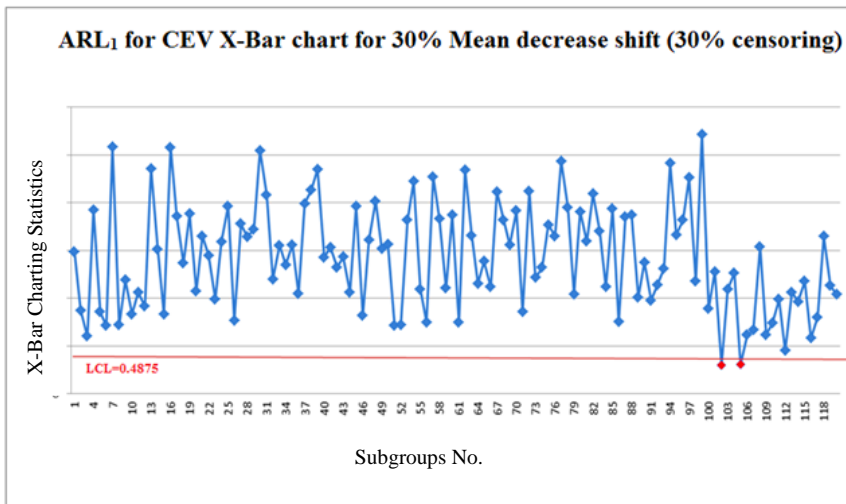
Figure 3: ARL$_1$ for CEV $\bar{X}$ Chart for 30% Mean Decrease Shift (30% Censoring)

5. Conclusions

In this article a Shewhart control chart based on the CEVs is used for monitoring the mean level of Gamma lifetimes with Type-I censoring. In the presence of Type-I censored data the existing/traditional control charts like and S control charts illustrate

large false alarm rate which results in low power. The advantage of the proposed method is that the practitioners can detect the change in process mean level by using proposed CEV Shewhart control chart when the data is censored using fixed censoring levels respectively (the detection of mean shifts in decreasing or in increasing order).

The comparison shows that the CEV Shewhart control charts outperform to traditional control charts in the presence of Type-I censored data. It is also observed that with the increase in censoring rates the results get more preferable for proposed CEV control charts.

The scope of the article may be extended to the Cumulative Sum (CUSUM) and Double EWMA charts for Gamma distributed quality characteristic(s) of interest. Also other distributional environments may be explored for the three control charting structures namely EWMA, CUSUM and Shewhart under censored process environment. Moreover, additional run rules schemes may be implemented with the said censored control structures for an enhanced monitoring of process parameters.

References

Acosta-Mejia, C. A. (1998). Monitoring reduction in variability with the range. *IIE Transactions, 30(*6), 515–523.

Antzoulakos, D. L., & Rakitzis, A. C. (2010). Runs rules schemes for monitoring process variability. *Journal of Applied Statistics*, *37* (7), 1231-1247.

Lu, W., & Tsai, T. R. (2008, June). Exponentially weighed moving average control chart for gamma distribution with type I censoring. *Innovative Computing Information and Control, 2008. ICICIC'08.* 3rd International Conference (pp. 128-128). IEEE.

Raza, S. M. M., Riaz, M., & Ali, S. (2015). On the performance of EWMA and DEWMA control charts for censored data. *Journal of the Chinese Institute of Engineers*, *38*(6), 714-722.

Raza, S. M. M., Riaz, M., & Ali, S. (2016). EWMA control chart for Poisson–Exponential lifetime distribution under type I censoring. *Quality and Reliability Engineering International*, *32*(3), 995-1005.

Roberts, S. W. (1959). Control Chart Tests Based on Geometric Moving Averages, *Technometrics*, *1*(3), 239-250.

Shewhart, W. A. (1931). *Economic control of quality of manufactured product*. ASQ Quality Press.

Steiner, S. H., & Mackay, R. J. (2000). Monitoring processes with highly censored data. *Journal of Quality Technology*, *32*(3), 199.

Steiner, S. H., & MacKay, R. J. (2001). Monitoring processes with data censored owing to competing risks by using exponentially weighted moving average control charts. *Journal of the Royal Statistical Society: Series C (Applied Statistics), 50*(3), 293-302.

Tsai, T. R., & Lin, C. C. (2009). The design of EWMA control chart for average with type-I censored data. *International Journal of Quality & Reliability Management*, *26*(4), 397-405.

Zhang, L., & Chen, G. (2004). EWMA charts for monitoring the mean of censored Weibull lifetimes. *Journal of Quality Technology*, *36*(3), 321-32