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The Valuation and Optimal Policies of Puttable Convertible Bonds

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Abstract

American-style convertible bonds commonly contain the put provision that allows the investors to put or sell their holdings to the issuer at preset prices and dates. The embedded put option includes a free boundary in addition to the conversion boundary. Because of the correlation of two moving boundaries with the convertible price, the valuation of puttable convertible bonds remains a classical problem in quantitative finance. This paper presents the valuation model of puttable convertible bonds under the Black-Scholes framework. We distinguish between the conventional pricing model and the current work by the realization of a jump in the put price across the hitting time. The jump condition permits the derivation of two recombining differential systems and we explore the impact of jump effect on the pricing dynamic of this innovative financial derivative.

***Keywords*:** Puttable convertible bonds, free boundary problem, jump conditions

AMS Classification: 91G20, 91G30, 91G80

Introduction

Puttable securities have been issued in various forms by corporations in recent years, often motivated by the need to protect investors against a significant decline in the value of derivatives. The put provision in puttable convertible bonds (PCB) gives the investors the right to sell their investments back to the issuer at prespecified prices and dates and it helps to mitigate the effects of security mispricing (Brick, Palmon & Patro., 2015). The put clause protects the investors against any or all types of risk factors by lowering the exercise boundary in the stock price, below which it is optimal for the bondholder to exercise the put option (Błach & Łukasik, 2017; Elkamhi, Ericsson & Wang, 2012). This is in

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addition to the upper exercise boundary, above which it is optimal for the bondholder to exercise the conversion feature. Because of the downside risk protection, PCB has become the most successful financial innovation in the convertible bond market during the last few years (Chemmanur & Simonyan, 2010).

The predominant methods for pricing convertibles without put option are analytical methods (Nyborg, 2006; Zhu, 2006) and numerical methods (Yang, Yu, Xu & Fan, 2018; Barone-Adesi, Bermudez & Hatgioannides, 2003). However, a significant proportion of convertible bonds issued after the celebrated work of McConnell and Schwartz (1986) has only a put option (Grimwood & Hodges, 2002). While it is interesting to understand what drives the issuance of PCB, little attention has been devoted to its valuation. Most recent works on PCB have focused on the qualitative rationales for its issuance (Brick et al., 2015; Chemmanur & Simonyan, 2010).

PCB, being an American-style security, can be redeemed by the bondholder before maturity either by exercising the conversion or put privilege. The possibility of early conversion leads to a free boundary S_{f1}, separating the region where it is optimal to hold the bond from the region where conversion is optimal. On the other hand, a rational investor seeks to maximize the value of PCB by his put right. Unless the stock price rises to a significant premium over the conversion price, investors may be better off exercising the put option than waiting to convert to equity. Such a financial decision creates an additional moving boundary S_{f2}. Apparently, the essential difficulty in valuing PCB lies in the fact that there are two unknown moving boundaries comprising the optimal exercise prices at the varying value of time which must be found as part of the solution.

Economically, the exact location of the unknown free boundaries (critical asset prices) is crucial to the investors because most of the arbitrage opportunities are expected near the free regions. However, the conventional pricing methods, that is the PDE-based numerical methods and Monte Carlo simulation method, are less dependable due to the discontinuity along both moving boundaries. Thus, it is desirable to extend the current literature by further exploring some PCB properties that may help to put its valuation in the proper context. In the subsequent sections, we review the free boundary problem of a PCB and explore some of its properties including the analysis of the behaviors of optimal exercise prices. The paper ends with the summary and concluding remarks in the last section.

2. Reduced-form Model of PCB

Before going into details, we shall make some necessary assumptions including but not limited to the usual Black-Scholes (1973) assumptions. However, some of these assumptions may be relaxed later to reflect advances in PCB trading.

- The capital market is perfect; there is no transaction cost, no taxes, and the information is homogeneous.
- Trading takes place continuously in time and no restrictions exist against borrowing or short sales.
- The investors are rational, they prefer wealth to loss and will always elect to invest in the dominant assets.
- The underlying stock price follows a lognormal diffusion process with constant volatility.
- The term structure of interest rate is flat and non-stochastic. The instantaneous compounding risk-free of interest is r.
- The dividend payment to the stockholders is continuous at a rate of D_0, and zero-coupon pay on bonds.
- The PCB indenture allows for only `block conversion' of the bonds into issuing stocks and there is no senior debt.
- Both the conversion and the put features are of American type which means that they can be exercised at any time up to bond maturity.
- Credit risk (default) is neglected.

Some of our assumptions demand further comments. Block conversion' suggests that all investors will convert at the same time when the condition of the optimal conversion strategy is met. However, under certain financial provisions, such as monopoly on convertible holdings or when there is additional subordinated debt, the conversion of one convertible can increase the value of those left unconverted (Dutordoir, Lewis, Seward & Veld, 2014). In such scenarios, it would seem plausible to make allowance for sequential conversion strategy. To simplify the analysis, we assume that the possession of all convertibles is diffused; each investor is a price-taker and that there is no debt junior to convertible bonds. Thus, only block conversion is considered. Furthermore, it is worth mentioning that the case of convertible bonds with default risk has been studied extensively in the literature. Considering credit risk introduces additional complexity, and most of the recent research on convertible bonds is concerned with the interplay of credit and equity risk on their valuation.

2.1. Problem Formulation

Let $\boldsymbol{V(S,t)}$ *denote the value of a PCB, which is a* measurable function of the underlying stock price $\boldsymbol{S}$ and time $\boldsymbol{t}$. Under the risk-neutral measure, the lognormal diffusion process that the stock price follows is given by the equation (1).

$$\mathrm{dS} = (\mu - \mathrm{D}_0)\mathrm{Sdt} + \sigma \mathrm{Sd}\omega \tag{1}$$

where $d\omega$ is a standard Wiener process, μ is the drift rate and D_0 is the continuous dividend rate.

Following the no-arbitrage argument, it can be shown that the fair value of convertible bonds (CB) satisfies the Black-Scholes equation,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (\mu - D_0)S\frac{\partial V}{\partial S} - rV = 0,\ \ 0 < S < S_{f1},\ t > 0 \tag{2}$$

Where S_{f1} is the critical level of the asset price at which it is optimal from the holder's point of view to convert the bonds into underlying stocks. To uniquely determine the fair price of the bond and optimal exercise prices, equation (2) needs to be solved with a set of appropriate boundary conditions.

2.2. Boundary Conditions

PCB is a hybrid financial derivative with the properties of both bond and option, hence both points of view provide a boundary condition for its valuation. At expiry, the holder of a PCB can decide to redeem it at face value $\boldsymbol{Z}$ or convert it to n units of underlying stock, '$\boldsymbol{n}$' being the number of shares of the issuer's common stock into which convertibles can be converted (also known as conversion ratio). Therefore, the payoff at maturity reads.

$$V(S,T) = \max(nS, Z) \tag{3}$$

The premature termination of the contract is at the discretion of the bondholders who have both the put and conversion privileges. The conversion right allows the investor to convert the bonds into underlying stocks any time prior to and on the expiry date. Following the argument of McConnell and Schwartz (1986), the value of the convertibles always stay equal to or above the conversion value (also known as parity), that is, the value of the investment is worth at least as much as the conversion value. We represent this constraint by the equation (4).

$$V(S,t) \geq nS \ for\ 0 \leq S \leq S_{f1},\ t \geq 0 \quad (4)$$

The conversion privilege creates an upper boundary. When the investors decide voluntarily to convert the bond into shares, the value of the bond equals the parity. Otherwise, there could be an arbitrage opportunity. The optimal conversion conditions imply that at each point in time *t*, there is a value of $S = S_{f1}(t)$ which marks the boundary between the holding region and the conversion region. Therefore, the boundary condition at $S = S_{f1}(t)$ is given by the equation (5).

$$V\left(S_{f1}(t), t\right) = nS_{f1}(t), \quad \frac{\partial V}{\partial S}\left(S_{f1}(t), t\right) = n \quad (5)$$

Since S is unknown in this region, a smooth-pasting condition is needed for determining the unknown boundary. In convertible trading, a possible scenario of condition (5) would be when PCB is *in-the-money*, that is, the underlying share price has increased greatly and is trading higher than the conversion price, then the price of PCB converges more and more towards conversion value. Let K denote the put price (assumed constant) and guaranteed repayment if the put right is exercised by the holder. As previously stated, the extra put clause reduces the investors' incentive distortions by allowing them to obtain a fair return if they observe that the firm is engaging in sub-optimal investment policies. Therefore, the bond stays alive only if its value is at least equal to the put value. This constraint is represented by the equation (6).

$$V(S,t) \geq K \ for\ 0 \leq S \leq S_{f2},\ t \geq 0 \quad (6)$$

Under the assumption of a complete capital market, finance theory predicts that the put right will be exercised optimally by the bondholder when the value of the convertible is equal to the put value. Contrary to the conversion strategy, which acts as an upper boundary, the optimal put strategy places a lower free boundary given by the equation (7).

$$V\left(S_{f2}(t), t\right) = K, \quad \frac{\partial V}{\partial S}\left(S_{f2}(t), t\right) = 0 \quad (7)$$

Financially, such a boundary condition implies that the convertible would be of the same value as the put value under adverse market conditions, such as falling of the stock price, rising issuer credit risk and interest rate. The put provision raises the *bond floor*, that is, the value of the convertibles if it were stripped of the possibility of converting into underlying shares. Apparently, convertible bonds with a put feature are worth more than those without the provision provided the put privilege is effective. However, if an issuer has a severe liquidity crisis, he may be incapable of paying for the bonds when the investors wish. Furthermore, the conditions (4) and (6) suggest that at each point in time there are in general two distinct stock prices where downside and upside constraints become binding. These limiting stock prices are unknown and must be determined as part of the solution. In fact, they are unknown boundaries beyond which the governing equation (2) does not apply.

The valuation problem of a PCB is now completely defined by a differential system composed of equations (2), (3), (5), and (7) on the domain $[0, S_{f1}] \times [0, T]$. (2). This can be written as a closed differential system as follows,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (\mu - D_0)S\frac{\partial V}{\partial S} - rV = 0, \qquad 0 < S < S_{f1}, \quad t > 0$$

$$\begin{aligned} V(S,T) &= \max(nS, Z) \\ V(S_{f1}(t), t) &= nS_{f1}(t) \\ \frac{\partial V}{\partial S}\left(S_{f1}(t), t\right) &= n \\ V(S_{f2}(t), t) &= \mathrm{K} \\ \frac{\partial V}{\partial S}\left(S_{f2}(t), t\right) &= 0 \end{aligned} \qquad (8)$$

The location of the optimal conversion boundary S_{f1} at expiry is given by the equation (9).

$$S_{f1}(T) = max\left(\frac{Z}{n}, \frac{\rho Z}{nD_0}\right) \qquad (9)$$

where ρ is the rate of continuous coupon payment on bond. Since $\rho = 0$ in the current paper, equation (9) can be written as $S_{f1}(T) = \frac{Z}{n}$.

3. Some Properties of the Puttable Convertible Bonds

In PCB trading, investors seek to maximize the value of their investment with the put privilege. Thus, it is optimal for the investor never to exercise the embedded put option when the convertibles' fair value is higher than the put value $\boldsymbol{K}$. The value $\boldsymbol{V}\ (\boldsymbol{S}, \boldsymbol{t})$ of PCB must satisfy the constraint (6). Equivalently, at any time $\boldsymbol{t}$, $\boldsymbol{K} \leq \boldsymbol{V}\ (\boldsymbol{S}, \boldsymbol{t})$. Now, we define the optimal exercise prices $\boldsymbol{S_{f1}(t)}$ and $\boldsymbol{S_{f2}(t)}$ as measurable functions of $\boldsymbol{S}$ and $\boldsymbol{t}$ by the equations.

$$S_{f1}(t) = \{(S,t)|V(S,t) = nS\}$$

$$S_{f2}(t) = \{(S,t)|V(S,t) = nS\}$$

In the absence of counterparty risk, for the optimal put boundary we have $S_{f2}(t) = [O, \bar{S}_{f2}]$, where $\bar{S}_{f2} = \sup\{S|S \in S_{f2}\}$.

Let t_c be the *hitting time* of the critical put price S_{f2}; t_c is the corresponding value of time t at which the premium or insurance associated with the put option becomes worthless. Thus, at the instant t_c and beyond, the put option becomes inactive. Clearly, t_c is the minimum value of time satisfying $V\ (0, t) \geq K$. The term $Ze^{-r(T-t)}$, which denotes the discounted cash flows coming from the convertible, becomes the effective lower boundary at instant $t > t_c$. Hence, it will be sub-optimal for an investor to put the convertible after t_c because the discounted cash flow at that instant is higher than the put value. Following the optimal strategy of PCB, at instant t_c the value of the convertible is equal to the put value. Therefore, one could determine t_c explicitly as

$$t_c = T - \frac{1}{r}\ln\frac{Z}{K} \qquad (10)$$

It is seen in equation (10) that t_c is sensitive to interest rate, the tenor of the contract and the ratio of cash flows coming from the convertible. When $Z/K \geq 1$, the put clause is rendered redundant and the bond behaves just like the standard CB. However, when $Z/\ K < 1$, t_c exceeds T, which is financially impracticable since put price is worth at most equal to the principal. In PCB trading, the new-found equation

(10) would not only be important for numerical computation but also has some financial advantages in that PCB under different bond parameters could be easily compared. The longer t_c becomes, the more valuable the embedded put option is.

a) Proposition 1. *The value of a PCB,* $V\ (S,t)$ *satisfies the inequalities* $V\ (S,t)\ \geq\ \bar{V}\ (S,t)$ *for all* S *and* t *where* $\bar{V}\ (S,t)$ *is the value of a non-PCB (convertible bonds without put feature).*

Proof. The indenture of an ordinary CB entitles the investors only to the incentive from converting the convertibles to stock before maturity, whereas, that of PCB entitles the investors to incentives from both put and conversion rights. Then $V\ (S,t)\ \geq\ \bar{V}\ (S,t)$ can be immediately followed with the constraint $V\ (S,t)\ \geq\ nS$ for a non-PCB and $V\ (S,t)\ \geq\ max(K,nS)$ for a PCB.

b) Proposition 2. *For the optimal put boundary,* $\bar{S}_{f2}$, *the relationship* $\bar{S}_{f2} < \frac{K}{n}$ *must hold.*

Proof. We shall assume that there exists an optimal put boundary, say $\tilde{S}$ such that $\tilde{S}\ \geq\ K/n\ \forall\ t$. This implies that when the stock price reach $\tilde{S}$, the put right will be exercised. From the definition of S_{f2}, when the put right is exercised optimally, we have $V\ (\tilde{S},t)\ =\ K$ for some $\tilde{S}\ \geq\ K/n$. On the other hand, if the put boundary K/n is chosen, CB will not be put and $V\ (\tilde{S},t)\ >\ K$ for $\tilde{S} \geq\ K/n$. The optimal put boundary $\tilde{S}$ does not result in the maximum value of CB. Hence, $\tilde{S}$ cannot be the optimal put boundary. This means that the optimal put boundary must satisfy $S_{f2}\ <\ K/n$.

c) Proposition 3. *At maturity, the critical asset price* S_{f1} *of the optimal conversion boundary of a PCB which satisfies* $S_{f1}(T) = max\left(\frac{Z}{n}, \frac{\rho Z}{nD_0}\right)$.

Proof. First, we need to incorporate coupon payment into equation (2). To achieve this, we define $Q\ =\ \rho Z$ as the total coupon payment and suppose that in every time interval $(t, t\ +\ dt)$ the bondholder will receive coupon payment amounting to Qdt. Then, we must add Qdt to the change in the value of the portfolio and equation (2) can be modified as follows,

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (\mu - D_0)S\frac{\partial V}{\partial S} - rV,\ 0 < S < S_{f1},\ \tau > 0 \qquad (11)$$

where $\tau = T - t$ is defined as the time of expiry. At expiration, for $S > Z/n$, we have $V(S,0) = nS_{f1}$. When $\tau \rightarrow 0^+$, and $S > Z/n$, by the continuity of PCB.

$$V(S, 0^+) = nS_{f1} \tag{12}$$

When the convertible is alive, it satisfies the governing equation (11). By substituting

(12) into (11), we have

$$\frac{\partial V}{\partial \tau}_{|\tau=0^+} = -nD_0S_{f1} + \rho Z \leq 0 \tag{13}$$

On the other hand, for $S < Z/n$, we are in the continuation region. At expiry, $V(S,0) = Z$. Where $\tau \rightarrow 0^+$, $V(S,0^+)$ will be determined by equation (11). As in the continuation region, the $V(S,\tau)$ is always above the conversion value nS. To keep it alive until expiry, we have

$$\frac{\partial V}{\partial \tau}_{|\tau=0^+} \geq 0 \tag{14}$$

The value of S at which $\frac{\partial V}{\partial \tau}_{|\tau=0^+}$ changes sign satisfies $S_{f1} = \frac{\rho Z}{nD_0}$. As $\frac{\rho Z}{nD_0}$ lies in the interval $S > Z/n$ only when $\rho > D_0$, if $\rho \leq D_0$, the changing sign point will become $S_{f1} = Z/n$. The optimal conversion price $S_{f1}(0^+)$ is given by the underlying value at which $\frac{\partial V}{\partial \tau}_{|\tau=0^+}$ changes sign. We then obtain $S_{f1}(0^+)$ by solving the following equation.

$$S_{f1}(0^+) = max\left(\frac{Z}{n}, \frac{\rho Z}{nD_0}\right) \tag{15}$$

3.1. Remarks

It should be remarked that the proof of proposition (3) exists in literature about convertibles. It is, however, reproduced here for completeness of the current work. Also, it should be reminded that ρ = 0 in the current paper.

To this end, it is worth noting that the optimal put and conversion boundaries are applicable in exactly opposite scenarios (that is decreasing and increasing claim value). The interactions between them are yet to be fully examined. Johnson (2003) quantitatively showed that both boundaries are independent at maturity and their mutual interaction is not significant over the life of the bond. Although his first assertion is obvious since the optimal put price is zero at the instant t_c and beyond, however, the results in the subsequent sections do not support his second claim. These and some interesting yet not-well recognized properties of PCB will be examined subsequently.

3.2. Effective and Non-Effective Put Privilege

Imagine that when the put price K is sufficiently low, say $K \leq Ze^{-rT}$ where T denotes the tenor of the contract, there is no incentive for the bondholder to redeem the convertibles under adverse market conditions so that the put clause is rendered redundant. In such a scenario, the embedded put privilege is non-effective and PCB simply resembles the usual standard CB. On the other hand, when K is sufficiently high one may foresee the early redemption of PCB prior to conversion. However, in financial practice, put price is always less than the face value, that is, $K < Z$. If K were to greater than Z, then Z would be rendered worthless since an investor can always put the convertibles prior to maturity to avoid receiving Z. Interestingly, the put value could be sufficiently high and yet the additional privilege remains non-effective.

A possible scenario would be when PCB is issued by firms with positive private information (the firms assess a lower probability of their put option being exercised as compared to overvalued firms). According to Chemmanur and Simonyan (2010), the firm can have information superior to investors about future earnings and cash flows (and, as such, about the intrinsic value of its equity). In this case, a firm whose equity is currently undervalued relative to its intrinsic value is more likely to bundle a put option when issuing a convertible debt. Though the put value could be sufficiently high, the investors are not likely to put the bond since there is an upside potential in equity. Thus, the put right becomes non-effective.

4. Behaviors of the Optimal Exercise Boundaries

4.1. Optimal Conversion Boundary

The optimal conversion boundary S_{f1}, which comprises the critical asset price for optimal conversion at varying time, has been studied in the literature (Zhu, 2006; Zhu & Zhang, 2012). S_{f1} decreases monotonically with time, more rapidly near expiry, and finally approaches the conversion price at expiry. Close to the contract expiration date, S_{f1} changes drastically which results in a very large gradient. For a zero-coupon PCB, the dividend on the underlying asset and unfavorable change in conversion terms are the major factors affecting the optimal exercise price, S_{f1}. If $D_0 \rightarrow 0$, PCB is worth more alive than conversion, in fact, the conversion right becomes worthless and $S_{f1} \rightarrow \infty$. Financially, early conversion leads to the loss of the insurance value associated with the embedded conversion privilege but results in no gain from the earlier possession of shares. However, if we relax our earlier assumption on the conversion ratio, any unfavorable change in conversion terms can be expressed in terms of reduced parity. A reduction in the conversion ratio may cause discontinuity in the free boundary. Consequently, it is optimal for investors to convert voluntarily before the change of conversion terms. On the other hand, in a coupon payment PCB, the cash flow advantage is crucial to an investor in decision making and this may subsequently affect the behavior of the critical conversion price. When the coupon earned on the bond is much higher than the dividend on the shares, it may be sub-optimal for an investor to convert prior to maturity.

Proposition. The underlying asset of a zero-coupon PCB can be traded for conversion value at expiry, that is, $\mathrm{S_{f1}(T) = Z/n}$

Proof. The prove of proposition (4) follows from the proof of proposition (3).

4.2. Optimal Put Boundary

The optimal put boundary, S_{f2}, consist of time varying put price. Studies involving the behavior of optimal put boundary are quite limited in literature. Thus, this work may form the basis for further studies in this area. Like the S_{f1}, S_{f2} decreases monotonically with time, more rapidly near t_c, and finally approaches zero at t_c. It should be noted that, for all $t < t_c$, S_{f2} decreases with time and its non-zero value is to be found as part of the solution, and for $t \geq t_c$, $S_{f2} = 0$. Obviously, S_{f2} drops to zero at t_c and such a drop constitutes a jump in the critical put price. When such behavior occurs in financial literatures, it always leads to an interesting academic exercise. Hence, it is expedient to investigate the effect(s) of the inevitable jump on the pricing dynamic of PCB.

4.3. Jump Conditions for Puttable Convertible Bonds

The jump conditions arise when there is a discontinuous change in one of the independent variables affecting the claim value of derivative security. In the problem defined above, jump condition arises as a result of discontinuous change in put price. The condition relates the value of PCB across t_c. Suppose we denote t_c^- and t_c^+ as the instants right before and after t_c, as mentioned previously, the optimal put price S_{f2} experiences a jump in value across t_c. The no-arbitrary pricing theory however requires that the claim's value should remain continuous. Therefore, for any fixed S_{f2}, the value of PCB remains continuous across t_c, and we have the equation

$$V\,(S, t_c^-\,) \;=\; V\,(S, t_c^+\,) \tag{16}$$

Following the equation (16) and using the value of t_c obtained from equation (10), we can effectively solve the pricing problem of PCB. To achieve this, consider dividing the domain $[0, T]$ into two parts, $[0, t_c^-\,]$ and $[t_c^+\,, T]$. Between the two-time sub-domain $V\,(S, t)$ is constant. Hence, it is a reasonable idea to solve the differential system (8) in two stages. The first stage in $[t_c^+\,, T]$ and the second stage in $[0, t_c^-]$. The solution obtained in the first stage will provide the terminal condition for the PDE system and in the second stage through the jump condition (16). Therefore, the free boundary problem of PCB can be expressed equivalently by recombining two PDE systems as follows,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (\mu - D_0) S \frac{\partial V}{\partial S} - rV = 0, \qquad 0 < S < S_{f1},\quad t > t_c$$

$$\begin{aligned} V(S, T) &= \max(nS, Z) \\ V(S_{f1}(t)\,, t) &= nS_{f1}(t) \\ \frac{\partial V}{\partial S}\left(S_{f1}(t)\,, t\right) &= n \\ \lim_{S\to 0} V\,(S, t) &= Ze^{-r(T-t)} \end{aligned} \tag{17}$$

and

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (\mu - D_0) S \frac{\partial V}{\partial S} - rV = 0, \qquad 0 < S < S_{f1},\quad t < t_c$$

$$V(S, t_c^-) = V(S, t_c^+)$$
$$V(S_{f1}(t), t) = nS_{f1}(t)$$
$$\frac{\partial V}{\partial S}\left(S_{f1}(t), t\right) = n \qquad (18)$$
$$V(S_{f2}(t), t) = \mathrm{K}$$
$$\frac{\partial V}{\partial S}\left(S_{f2}(t), t\right) = 0$$

Clearly, equation (17) shows that at the given time to expiry, the two-unknown boundaries are independent and there is no significant relationship between S_{f1} and S_{f2} for any $t \in [t_c^+, T]$. These findings agree with the empirical analysis in literature. However, for equation (18), the numerical result obtained from (17) becomes its final condition through the jump condition (16). Also, the relationship between both boundaries is well pronounced because both boundaries are needed to properly close the PDE system.

5. Conclusions

We have presented two recombining PDE systems for pricing PCB. Our findings in this work are of great practical significance in quantitative finance as the results form the basis for developing the popular integral equation solution approaches. Moreover, with these findings the asymptotic behavior of the two optimal exercise boundaries near redemptions can be explored. Finally, our results reveal that the widely-used classical numerical methods might not price PCB efficiently because of the jump effects. Thus, it is essential for the market practitioners and researchers to be aware of the jump associated with the put price of PCB. Subsequent research could explore the integral equations pricing methods and asymptotic behavior of the moving boundaries.

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